7th Annual Bergen County Academies Math Competition

Sixth Grade

Sunday, 18 October 2009

1 Contest

1. $1 + 0 + 1 + 2 + 1 + 5 + 1 + 9 + 1 + 14 + 1 + 20 + 1 + 27 + 1 + 35 + 1 + 44 + 1 + 54 =?$

   Solution 1: $1 + 0 + 1 + 2 + 1 + 5 + 1 + 9 + 1 + 14 + 1 + 20 + 1 + 27 + 1 + 35 + 1 + 44 + 1 + 54 = \boxed{220}$.

   Solution 2: Notice that we have 10 1's and another 10 terms which are in the following progression: 0, 2, 5, 9, 14, 20, ..., where the difference between successive terms increases by one. Now, these numbers are one less than the triangle numbers, so adding a one to successive terms gives the triangle number (1, 3, 6, 10, 15, ...); the sum of 10 of these terms is $\sum_{n=1}^{10} \binom{n+1}{2}$. From the hockey-stick rule, this is equal to $\binom{12}{3}$, which is $2 \cdot 11 \cdot 10 = \boxed{220}$.

2. Yi has to write an essay that is 1005 words long. He writes 10 words on the first day, 20 words on the second day, and 30 words on the third day. He has 10 days to finish the essay. Over the next seven days, what is the average number of words he will need to write per day in order to finish the essay?

   Solution: In the first three days, Yi writes a total of 60 words. So, he has 7 days and 945 words remaining. $\frac{945}{7} = \boxed{135}$.

3. A circle with circumference 12 has an inscribed square. Find the area of this square.

   Solution: If the circle has circumference 12, its radius is $\frac{12}{2\pi} = \frac{6}{\pi}$. This is one-half the diagonal of the square, so the square's side is $\frac{6}{\pi}\sqrt{2}$, and thus its area is $\boxed{\frac{72}{\pi^2}}$.

4. Kelly's favorite number is a positive integer less than 50. Both of the digits are odd. The tens digit is greater than the ones digit. What is Kelly's favorite number?

   Solution: Since the ten's digit cannot be 1 (as it must be greater than the one's digit), it must be 3. But the only odd number less than 3 is 1, so the answer is $\boxed{31}$.

5. If the ratio of the surface area to the volume of a sphere is 1:3, what is its radius?

   Solution: The volume of a sphere is $\frac{4}{3}\pi r^3$, the surface area $4\pi r^2$. So we simply need to find r such that $\frac{4}{3}\pi r^3 = 12\pi r^2$, which reduces to $\frac{1}{9}r = 1$, so $r = \boxed{9}$.

6. Let $f(x) = \frac{x^3-3x-9}{x^3-7x-6}$. Compute the sum of all distinct values of x for which $f(x) = 1$.

   Solution: For $f(x)$ to be one, the numerator and denominator must be equal. Thus, $x^3 - 3x - 9 = x^3 - 7x - 6$, which we can easily reduce to $4x = 3$, making x equal to $\boxed{\frac{4}{3}}$.

7. What is the largest prime factor of $34 \cdot 51 \cdot 68$?

   Solution: We can figure this out simply by factoring this number. That factorization is $(2 \cdot 17)(3 \cdot 17)(4 \cdot 17)$, from which we easily see that the largest prime factor is $\boxed{17}$.

8. There are fifty marbles in a jar. Three of them are white, and thirty-three of them are black. If I draw out a marble, what is the probability that it is neither white nor black?

   Solution: Since there are a total of 36 white-or-black marbles, there are 14 neither-black-nor-white marbles. Thus, the probability of drawing one is $\frac{14}{50} = \boxed{\dfrac{7}{25}}$.

9. Let $a \diamond b = \dfrac{a^2}{2} - b$. Evaluate $6 \diamond (4 \diamond 3)$.

   Solution: $4 \diamond 3 = \frac{16}{2} - 3 = 5$. $6 \diamond 5 = \frac{36}{2} - 5 = \boxed{13}$.

10. If Pavel travels along the path from $(0, 0)$ to $(12, 16)$ in 5 seconds, what is his average speed?

    Solution: From the Pythagorean theorem, the square of the distance traveled is $12^2 + 16^2 = 4^2(3^2 + 4^2) = 4^2 \cdot 5^2 = 20^2$. Thus Pavel travels 20 in 5 seconds, giving a total speed of $\frac{20}{5} = \boxed{4}$.

11. If someone submits n problems for the Bergen County Academies math competition, they receive $\left\lfloor \dfrac{n}{12} + 1 \right\rfloor$ community service hours, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. What is the minimum number of problems a single person needs to submit in order to receive seven hours?

    Solution: We need to solve the equation $\left\lfloor \dfrac{n}{12} + 1 \right\rfloor = 7$. We know that $\left\lfloor \dfrac{n}{12} + 1 \right\rfloor = \dfrac{n}{12} + 1$ when $\dfrac{n}{12} + 1$ is a whole number, i.e. n is a multiple of 12. Therefore, to solve the equation $\dfrac{n}{12} + 1 = 7$, we must subtract one from both sides and then multiply both sides by 12, resulting in $n = (7 - 1) \cdot 12 = \boxed{72}$.

12. A cylinder with integer radius and height has surface area 42π. Find the minimum possible height of the cylinder.

    Solution: We have the surface area of a cylinder as $2\pi r^2 + 2\pi hr$. For it to be equal to 42π, we need $r^2 + hr$ to be 21. $r^2 + hr$ factors into $r(h + r)$, and 21 into $3 \cdot 7$. Thus, r must be 3 and h must be $\boxed{4}$.

13. In thirty years, Bob's age will have doubled. What is Bob's age now?

    Solution: If Bob's age now is x, in thirty years it will be $x + 30$, equivalently $2x$. Thus, $x = \boxed{30}$.

14. Mike is thinking of four *different* positive integers. Their product is 36. What is their sum?

    Solution: $1 \cdot 2 \cdot 3 \cdot 6$ is the only such set of four integers. This can be verified by noting that the smallest five factors of 26 are 1, 2, 3, 4, 6. However, four cannot appear in any decomposition, as that would leave three distinct numbers multiplying to 9, which is impossible (either two 1's or two 3's would appear). So the only possible decomposition is $1 \cdot 2 \cdot 3 \cdot 6$, whose value for the sum is $\boxed{12}$.

15. Compute $i^{2008} - i^{2006}$, given $i = \sqrt{-1}$.

    Solution: $i^4 = (i^2)^2 = (-1)^2 = 1$. So $i^{2006} = (i^4)^{501} \cdot i^2 = 1^{501} \cdot -1 = -1$. $i^{2008} = i^{2006} i^2 = -1 \cdot -1 = 1$. So we're looking for $1 - (-1) = \boxed{2}$.

16. Rob swims two laps in a circular pool. He swims the first lap at a speed of 20 inches per second. He completes his second lap in 30 inches per second. What is his average speed of his swim, in inches per second?

    Solution: Let the total circumference of the pool be x inches. Then his first lap takes $\dfrac{x}{20}$ seconds and his second $\dfrac{x}{30}$. His total time is thus $\dfrac{5}{60}x$ and total distance $2x$. His average speed is thus $\dfrac{120}{5} = \boxed{24}$.

17. How many integers from 1 to 1000 are multiples of 39 but not 13?

    Solution: All multiples of 39 are multiples of thirteen, since if $n = 39k$, then $n = 13(3k)$. Thus the answer is $\boxed{0}$.

18. James was adding the numbers 1, 2, 3, ..., and when he reached a certain number, the sum was 1,000. However, when he was checking his work, he founds out that he counted one of the numbers twice. What was that number?

    Solution: We know that the sum of the first n numbers is $\frac{n(n+1)}{2}$. We want to find the largest such number less than 1000. n should thus be close to the square root of 2000, or about 45. After some experimentation, we find that the n we want is 44. $\frac{44 \cdot 45}{2} = 990$, so the number we counted twice was $1000 - 990 = \boxed{10}$.

19. A line with slope 1 is moved to the right 5 units, moved down 2 units, mirrored over the x-axis, then mirrored over the y-axis. What is the slope of the resulting line?

    Solution: The first two motions, translations, do not affect the slope of the line at all. Now, each of the reflections that follows changes the sign of the slope, because it changes the sign of either the "rise" or the "run" of the line. But since we changed the sign of the slope twice, it is still $\boxed{1}$.

20. How many values of n are there such that $\dfrac{50}{n}$ is an integer?

    Solution: Clearly $\frac{50}{n}$ is an integer whenever n is a factor of 50, so we must count the number of factors of 50. We factor: $50 = 2 \cdot 5^2$, and each factor corresponds to a way to choose an exponent for 2 and an exponent for 5. We have 2 ways to choose the exponent for 2 and 3 ways to choose an exponent for 5, so we have $2 \cdot 3 = 6$ positive factors of 50. These positive factors, as well as their negatives, will make $\frac{50}{n}$ an integer. This means that the total number of such n is $2 \cdot 6 = \boxed{12}$.

21. If I roll two fair six-sided dice, what is the probability that the sum of the numbers on the tops is 8?

    Solution: In order for the sum to be 8, the numbers on the top of the two dice have to be one of $(2,6)$, $(3,5)$, $(4,4)$, $(5,3)$, or $(6,2)$. This gives five out of 36 total pairs, for an answer of $\boxed{\dfrac{5}{36}}$.

22. There are two colors of magic blobs: red blobs and blue blobs. Every minute, a red blob divides into two red blobs of equal volume and a blue blob divides into three blobs of equal volume: one red and two blue. If I start with a blue blob of volume 1, what volume of red blobs will I have in four minutes?

    Solution: Every minute, for every unit of volume I had before, I now have $\dfrac{2}{3}$ as much blob as we had before. So, after four iterations, we have $\dfrac{16}{81}$ as much volume as we started with. But since we started with 1 unit of volume, we end with $\boxed{\dfrac{16}{81}}$.

23. There are two numbers that satisfy the equation $(2x+3)(x-6)+(x-4)(2x+3) = 0$. What is their sum?

    Solution: We can combine the above equations as $(2x+3)(2x-10) = 0$. We can now see the two solutions for x: $x = 5$ or $x = -\dfrac{3}{2}$. The sum of these two is $\boxed{\dfrac{3}{2}}$.

24. At the academies, there is a game known as Mathathon. In Mathathon, points are scored by doing either "Picky Problems" or "Cute Questions." Each cute question is worth nine points and each picky problem is worth five points. Assuming there are an infinite number of both types of problems, and problems can only be scored in whole number amounts, how many scores are unattainable?

Solution: We can make a list of scores we can actually get: 5, 9, 10, 14, 15, 18, 19, 20, 23, 24, 25, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, ... Since its possible to achieve 27 to 31, we can achieve any number by adding 5 to one of those. Thus, the scores that are unattainable are 1, 2, 3, 4, 6, 7, 8, 11, 12, 13, 16, 17, 21, 22, 26, for a total of $\boxed{15}$.

25. Hannah can build a house in twenty-four hours. Beth can build the same house in forty-eight hours. If Beth starts building the house and Hannah starts helping her a day later, in how many days will the house be built?

    Solution: Let's consider the speed with which Hannah works. She builds one house in exactly one day, so we can say that she builds one house per day. Beth builds one house every two days, or one half of a house per day. Together, they thus build $\frac{3}{2}$ houses per day, so to build one half of a house (which is left over after day 1), they need $\frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$ days. But then, to build the house they need a total of $1 + \frac{1}{3} = \boxed{\frac{4}{3}}$ days.

26. Let $a\%b$ be the remainder when a is divided by b. What is (((((127%64)%32)%16)%8)%4)%2?

    Solution: 127%64 is an odd number, and it will stay an odd number until the end because an odd number % an even number is an odd number. Since the last function is an odd number % 2, the answer must be $\boxed{1}$ because that is the only odd number that can be a remainder when dividing by 2.

27. The formula for converting Fahrenheit to Celsius temperatures is $C = \frac{5}{9}(F - 32)$. Find the number of integer Fahrenheit temperatures between 0^o and 100^o that corresponds to an integer number of Celsius degrees.

    Solution: For $\frac{5}{9}(F-32)$ to be an integer, we must have that 9 divides $F-32$. So, F can be $5, 14, 23, \dots$ or 95. There are 11 numbers in the list, so our answer is $\boxed{11}$.

28. The area of circle O is 4 square meters. If the diameter of O is doubled, what is the area of O in square meters?

    Solution: If the diameter doubles, then the area quadruples because 2 times 2 is 4, and the radius is squared in the circle area formula. Thus the new area is $\boxed{16}$ square meters.

29. In a Computer Science AB class, Watson argues with Pavel for 68% of the class time and with Dr. Nevard for 17% of the class time, but he never argues with both at the same time. The rest of the time, Watson does not argue with anyone. If Watson is not arguing with anyone for m% of the class, find m.

    Since Watson is arguing with someone for exactly $68 + 17 = 85$% of the time, he is not arguing with anyone for $\boxed{15}$% of the time.

30. Jeff wants to order an appetizer and a drink at Boston Market. If Boston Market had one more appetizer, Jeff would have twenty more combinations to choose from. If Boston Market removed a drink, Jeff would have eight fewer choices. How many meals could Jeff choose from originally?

    Solution: Let A and D represent the number of appetizers and drinks available. $20 + AD = (A+1)D$, and $-8 + AD = A(D - 1)$, From these equations we see that there are 20 drinks and 8 appetizers available, and there are $\boxed{160}$ combinations for Jeff to choose from.

31. Kamran flips a quarter four times. What is the probability that he gets at least two heads?

    Solution: There are 16 possibilities. $6 + 4 + 1$ of these have at least 2 heads. Therefore the probability is $\boxed{\frac{11}{16}}$.

32. Evaluate $\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{2^3}+\dfrac{1}{3^3}+\cdots$

Solution: There are two geometric sequences. $\dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\cdots=1$ and $\dfrac{1}{3}+\dfrac{1}{3^2}+\dfrac{1}{3^3}+\cdots=\dfrac{\frac{1}{3}}{1-\frac{1}{3}}=\dfrac{1}{2}$.

So the answer is $\boxed{\dfrac{3}{2}}$.

33. Find the area of the region in the xy-plane that satisfies $y>|x|$ and $y<5-|x|$.

Solution: $y>|x|$ is a graph of $y=x$ for $x>0$ and $y=-x$ for $x\le 0$. If we shift this up by 5 and rotate it 180 degrees. The area is a square with diagonal length 5. Therefore its side length is $\frac{5}{\sqrt{2}}$.

Square this to get an area of $\boxed{\dfrac{25}{2}}$.

34. There is a class of fourteen people. Suppose that the students are giving a presentation on math problems, so that on each of the first three days, four students present, and on the fourth day, two students present. If the day each student is assigned to present is chosen randomly, find the probability that Michael will present on the last day.

Solution: There are two presentations out of fourteen on the last day. Therefore the probability is $\frac{2}{14}=\boxed{\dfrac{1}{7}}$.

35. What is 70% of 70%? Express your answer as a percent.

Solution: 70% of 70% is .7 times .7 = .49 = $\boxed{49\%}$.

36. Three fifth graders running at different speeds are racing around a circular tower. The fastest one can run around the tower in 5 minutes. The slowest one can run around the tower in 7 minutes. They start at the same point and they race around the tower in the same direction. After 35 minutes, they are all at the starting point. How long does it take for the second fastest runner to run around the tower?

Solution: The fastest runner has run 7 laps. The slowest runner has run 5 laps. Thus the middle-speed runner has run 6 laps in 35 minutes for a rate of $\boxed{\dfrac{35}{6}}$ minutes per lap.

37. $2+\left|1-\left|\dfrac{7}{4}\right|\right|=?$

Solution: We can disregard the absolute value signs around the $\dfrac{7}{4}$. $2+\left|1-\dfrac{7}{4}\right|=2+\dfrac{3}{4}=\boxed{\dfrac{11}{4}}$.

38. A $3\times 3\times 3$ cube made up of twenty-seven $1\times 1\times 1$ cubelets has all of its faces painted red. The eight corner cubelets are removed. Find the difference in the surface areas of the cube before and after the cubelets were removed.

Solution: There are six directions from which to examine the cube: front, back, right, left, top, bottom. Either way, we see a 3x3 square from each angle. Therefore the surface areas before and after are 54. There is no difference, hence $\boxed{0}$ is our answer.

39. Compute the larger root of $x^2=x+1$.

Solution: We have to solve $x^2-x-1=0$. The quadratic formula shows that the greater of the roots is $\dfrac{1+\sqrt{1+4}}{2}=\boxed{\dfrac{1+\sqrt{5}}{2}}$. Note that this is the famous golden ratio, φ.

40. Mike is preparing for the USA Math Olympiad. If he solves p problems on a certain day, then he will do $2 \cdot p + 1$ problems the next day. If Mike solves one problem the first day, how many problems will he do on the tenth day?

Solution 1: The sequence is 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023. So the answer is $\boxed{1023}$.

Solution 2: This can also be done by noting that adding one to all these numbers forms a sequence of the powers of 2. In particular, $2(2^p - 1) + 1 = 2^{p+1} - 1$. So all the members of this sequence are one less than a power of 2, because we started with a number that is one less than a power of 2. The number of problems Mike does on the tenth day is $2^{10} - 1 = 1024 - 1 = \boxed{1023}$.

41. Let $A = (0, 0)$, $B = (4, 0)$, $C = (0, 3)$, and $D = (5, 5)$. What is the length of the shortest path which passes through the points A, B, C, and D?

Solution: The shortest path between any two points is a straight line; therefore, the shortest path between a collection of points is a set of line segments. Since there are many such paths, we will first construct a table of distances that we can then analyze.

Line Segment	Distance
$\overrightarrow{AB}$, $\overrightarrow{BA}$	4
$\overrightarrow{AC}$, $\overrightarrow{CA}$	3
$\overrightarrow{AD}$, $\overrightarrow{DA}$	$\sqrt{5^2 + 5^2} = \sqrt{50}$
$\overrightarrow{BC}$, $\overrightarrow{CB}$	$\sqrt{3^2 + 4^2} = \sqrt{25} = 5$
$\overrightarrow{BD}$, $\overrightarrow{DB}$	$\sqrt{(5-4)^2 + 5^2} = \sqrt{26}$
$\overrightarrow{CD}$, $\overrightarrow{DC}$	$\sqrt{5^2 + (5-3)^2} = \sqrt{29}$

From this list of 12 directed line segments, we can construct a list of 24 possible paths.

Path	Length
#1 $\overrightarrow{ABCD}$, $\overrightarrow{DCBA}$	$4 + 5 + \sqrt{29} = 9 + \sqrt{29}$
#2 $\overrightarrow{ABDC}$, $\overrightarrow{CDBA}$	$4 + \sqrt{26} + \sqrt{29}$
#3 $\overrightarrow{ACBD}$, $\overrightarrow{DBCA}$	$3 + 5 + \sqrt{26} = 8 + \sqrt{26}$
#4 $\overrightarrow{ACDB}$, $\overrightarrow{BDCA}$	$3 + \sqrt{29} + \sqrt{26}$
#5 $\overrightarrow{ADBC}$, $\overrightarrow{CBDA}$	$\sqrt{50} + \sqrt{26} + 5$
#6 $\overrightarrow{ADCB}$, $\overrightarrow{BCDA}$	$\sqrt{50} + \sqrt{29} + 5$
#7 $\overrightarrow{BACD}$, $\overrightarrow{DCAB}$	$4 + 3 + \sqrt{29} = 7 + \sqrt{29}$
#8 $\overrightarrow{BADC}$, $\overrightarrow{CDAB}$	$4 + \sqrt{50} + \sqrt{29}$
#9 $\overrightarrow{BCAD}$, $\overrightarrow{DACB}$	$5 + 3 + \sqrt{50} = 8 + \sqrt{50}$
#10 $\overrightarrow{BDAC}$, $\overrightarrow{CADB}$	$\sqrt{26} + \sqrt{50} + 3$
#11 $\overrightarrow{CABD}$, $\overrightarrow{DBAC}$	$3 + 4 + \sqrt{26} = 7 + \sqrt{26}$
#12 $\overrightarrow{CBAD}$, $\overrightarrow{DABC}$	$5 + 4 + \sqrt{50} = 9 + \sqrt{50}$

Now we must eliminate paths which cannot possibly be minimal. Since $5 = \sqrt{25} < \sqrt{26}$, $4 + \sqrt{26} > 4 + 5 = 9$, so $5 = \sqrt{25} < \sqrt{26}$, $4 + \sqrt{26} + \sqrt{29} > 9 + \sqrt{29}$, so we can eliminate path #2 and keep path #1. Similarly, since $5 < \sqrt{29}$, $5 + (3 + \sqrt{29}) < \sqrt{29} + (3 + \sqrt{29})$, so we eliminate path #4 and keep #3. $\sqrt{26} < \sqrt{29}$, so path #5 is shorter than path #6. $7 + \sqrt{29} - (4 + \sqrt{50} + \sqrt{29}) = 3 - \sqrt{50} = \sqrt{9} - \sqrt{50} < 0$, so path #7 is shorter than #8. Similarly, #9 is shorter than #10. With these eliminations, we are left with the following table.

Path	Length
#1 $\overrightarrow{ABCD}$, $\overrightarrow{DCBA}$	$9+\sqrt{29}$
#3 $\overrightarrow{ACBD}$, $\overrightarrow{DBCA}$	$8+\sqrt{26}$
#5 $\overrightarrow{ADBC}$, $\overrightarrow{CBDA}$	$\sqrt{50}+\sqrt{26}+5$
#7 $\overrightarrow{BACD}$, $\overrightarrow{DCAB}$	$7+\sqrt{29}$
#9 $\overrightarrow{BCAD}$, $\overrightarrow{DACB}$	$8+\sqrt{50}$
#11 $\overrightarrow{CABD}$, $\overrightarrow{DBAC}$	$7+\sqrt{26}$
#12 $\overrightarrow{CBAD}$, $\overrightarrow{DABC}$	$9+\sqrt{50}$

Comparing #3 and #11, we can immediately remomve #3; by comparing #1 and #7, we can remove #1. We can also remove #12 and leave #9. #7 is also clearly larger than #11, so we can remove it.

Path	Length
#5 $\overrightarrow{ADBC}$, $\overrightarrow{CBDA}$	$\sqrt{50}+\sqrt{26}+5$
#9 $\overrightarrow{BCAD}$, $\overrightarrow{DACB}$	$8+\sqrt{50}$
#11 $\overrightarrow{CABD}$, $\overrightarrow{DBAC}$	$7+\sqrt{26}$

Now we just need to do two more comparisons to find the minimum path. Subtracting #5 from #9 gives $8+\sqrt{50}-(\sqrt{50}+\sqrt{26}+5)=3-\sqrt{26}=\sqrt{9}-\sqrt{26}<0$, so #9 is smaller than #5. Our final comparison is between #9 and #11. Subtracting #11 from #9 gives $8+\sqrt{50}-(7+\sqrt{26})=1+\sqrt{50}-\sqrt{26}$. Since $\sqrt{50}\approx 7$ and $\sqrt{26}\approx 5$, we can approximate this as $1+7-5=3>0$. This means that #11 is the smallest path, and the minimum distance is $\boxed{7+\sqrt{26}}$.

42. The math team consumes 120 cans of Coca Cola and 199 cans of Canada Dry. Each math team member drinks at least one and at most three cans of Coca Cola as well as at least two and at most five cans of Canada Dry. If the math team has at least m and at most M members, what is $M-m$?

    Solution: First, let us consider the number of people needed to consume the Coca Cola. The minimum is acheived if everyone drinks the maximum of three cans; thus, the least number of math team members needed to drink the Coca Cola is $\frac{120}{3}=40$ people. The maximum is acheived if everyone drinks the minimum of one can; thus, the least number is $\frac{120}{1}=120$ people.

    Next, let's consider the Canada Dry. The minimum is acheived when as many people as possible drink the maximum of five cans; however, since there are only 199 cans (which is one less than a multiple of 5), one person must drink 4 cans. This happens when 39 people drink 5 cans, and 1 person drinks 4 cans ($39\times 5+1\times 4=195+4=199$); so the minimum is 40 people. There cannot be any less than 40 people, because then someone would have to drink more than 5 cans, which is not allowed. Similarly, there can be at most 99 people; 98 of those people drink 2 cans, and one person drinks 3 ($98\times 2+1\times 3=196+3=199$).

    The minimum number of people to drink both the Coca Cola and the Canada Dry is $m=40$. The maximum number for Coca Cola is 120, and the maximum for Canada Dry is 99; since we cannot have 120 people drink the Canada Dry, the overall maximum must be $M=99$. Thus, $M-m=99-40=\boxed{59}$.

43. Two positive integers are relatively prime if they share no factors other than 1. How many positive integers less than 100 are relatively prime to 100?

    Solution: The 50 positive odd integers not exceeding 100 but greater than or equal to 1 are relatively prime to 2. Of these, 10 are divisible by 5, which would make them not relatively prime to 100. We can remove these. Our final answer is $\boxed{40}$.

44. John has 30π meters of wire. In meters, what is the largest possible area the wire can encompass if the two endpoints of the wire must touch?

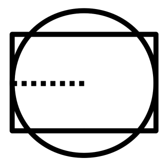
Solution: The largest possible area is a circle with perimeter 30π. So the radius is 15. The area is thus $\boxed{225\pi}$.

45. Kevin has three identical white marbles and four identical black marbles. He puts them in a bag and randomly draws them out one by one without replacing them. On a sheet of paper, he records the sequence of colors (ex: BBWWWBB). How many such sequences are possible?

    Solution: We can arrange the marbles in 7! ways. But we must correct our overcount of the 3 indentical black marbles and the 4 identical white marbles, which can be arranged in 3! and 4! ways respectively. Hence our answer is $\frac{7!}{3!4!} = 5 \cdot 7 = \boxed{35}$.

46. A rectangle with the area of 100 square feet has a circle inside it that lies tangent to 2 opposite sides and intersects the other two sides four times. If the area of this circle is 400π square feet, what is the length of the rectangle's shorter side?

    Solution:

    Since the circle's area is 400π, its radius is 20 (recall that $A = \pi r^2$, so $r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{400\pi}{\pi}} = \sqrt{400} = 20$). Because the circle is both intersects and is tangent to the rectangle, it must be tangent to the shorter sides of the rectangle (see the figure above); this means that the diameter of the circle ($2 \times r = 2 \times 20 = 40$) is equal to the length of the rectangle's longer side. Since the area of the rectangle is 100, the length of the shorter side is $\frac{100}{40} = \boxed{\frac{5}{2} = 2\frac{1}{2}}$.

47. An equilateral triangle is inscribed in a circle O with radius 6. Find the biggest possible radius of a circle that lies completely outside of the triangle and completely inside O.

    Solution: The circle's diameter is an extension of the triangle's median, and together they form the circle's diameter. Thus, if we know the length of the triangle's median, we can subtract this result from 12 to get our smaller circle's diameter. We know that the medians of a triangle intersect at a single point in $2:1$ proportion, so $\frac{2}{3}m = R$, where l is the triangle's median's length. Thus, we find that $m = 9$. Then, from $12 - m = 2r$, we find that the smaller circle's radius is $\boxed{\frac{3}{2}}$.

48. Brian has a collection of stones. Each weighs a whole number of pounds. By combining stones from the set, he can make any whole number of pounds from 1 to 63 pounds, inclusive. What is the fewest number of stones that Brian can have?

    Solution: Suppose we have n stones, and we wish to count the number of possible weights we can make using some or all of these stones. For each stone we have 2 choices: we either use the stone or we don't. Thus, we have at most 2^n different values we can make using n stones. Therefore, we must have at least 6 stones, since if $n < 6$ we have at most $2^n \leq 2^5 = 32$ choices, and we know we must have at least 63. To show that it is indeed possible to make all weights from 1 to 63 using 6 stones, consider the weights 1, 2, 4, 8, 16, and 32 pounds. Since every number between 1 and 63 has a base-2 representation of at most 6 digits, we can simply include the n-th stone when the n-th rightmost digit is 1. Thus, our final answer is $\boxed{\text{6 stones}}$.

49. Jordan thinks of a twelve-digit number such that the sum of any three consecutive digits is 17. If he divides number by 100, the remainder is 17. What is the leftmost digit?

Solution: The last two digits are $\underline{1}\,\underline{7}$ because that is the remainder when the number is divided by 100. Every subsequent digit to the left is obtained by subtracting the two digits to the right from 17. Hence we get 917917917917. The leftmost digit of this 12 digit number is $\boxed{9}$.

50. Find the smallest positive integer n such that 120 divides $14n$.

    Solution: We want $\frac{14n}{120} = \frac{7n}{60}$ to be an integer. Since 60 and 7 are relatively prime (they have a gcd of 1), 60 divides n. So the smallest possible value of n is $\boxed{60}$, because n cannot be 0.