1 Grade 7 Solutions

1. At the end, Sue is on rung number $9+6-2+3-9+11=18$. Since this is also the top rung, there must be $\boxed{18}$ rungs in total.

2. The answer is $\boxed{2}$.

3. We apply the divisibility rule for 3. The sum of the digits of 12345678, which is 36, is divisible by 3, so 12345678 is itself divisible by 3. Thus, it leaves a remainder of $\boxed{0}$ when divided by 3.

4. Notice that $22x+54=2(11x+27)$. The answer is thus $1234321\cdot 2=\boxed{2468642}$.

5. $2+2+4+8+16+32+64+128+256=\boxed{512}$

6. Let the palindrome be *abccba*. There are 9 choices for a (1-9), and 10 choices each for b and c (0-9). This gives a total of $9\cdot 10\cdot 10=\boxed{900}$ 6 -digit palindromes.

7. Split the 6 rabbits into two groups of three rabbits. In three minutes, each of the two groups of rabbits willl have eaten 3 carrots, so after $\boxed{\text{3 minutes}}$, they will have eaten 6 carrots in total.

8. $1+3+\cdots+45=(1+45)+(3+43)+\cdots+(21+25)+23=11(46)+23=506+23=\boxed{529}$.

9. They would have made $93500\cdot\dfrac{12}{11}=\boxed{102000}$ dollars.

10. There are 20 jelly beans in total, with 4 of them blue. The probability of the first not being blue is $\dfrac{16}{20}$, the probability of the second not being blue is $\dfrac{15}{19}$, and the probability of the third not being blue is $\dfrac{14}{18}$. Thus the probability of getting three non-blue jelly beans is $\dfrac{16}{20}\cdot\dfrac{15}{19}\cdot\dfrac{14}{18}=\boxed{\dfrac{28}{57}}$.

11. $g(1)=2$, $f(g(1))=f(2)=3$, $g(f(g(1))=g(3)=6$, $f(g(f(g(1))))=7$, $g(f(g(f(g(1)))))=14$, so $f(g(f(g(f(g(1))))=\boxed{15}$.

12. $2012=2\cdot 1006=2\cdot 2\cdot 503$. Since 503 is prime, our answer is just $2+503=\boxed{505}$.

13. Each cycle of the pattern consists of 3 forward steps and then 1 back step, which is a net gain of 2 forward steps. After 99 cycles, or 99 backward steps, the baby chick has made a net of $99\cdot 2=198$ steps, making it 2 steps away from the traffic light. Therefore, the next step forward will bring it to its destination, so it will have made a total of $\boxed{99}$ backward steps.

14. After every 20 minutes, the number of parasprites doubles: each of the original parasprites remains, and each of them spawns a new parasprite. Since six sets of 20 minutes pass in two hours, there will be a total of $2^6=\boxed{64}$ parasprites after 2 hours.

15. Since x^2y is constant for all values of x and y, we have that x^2y is always equal to $6^2\cdot 4=144$. Therefore, we need y such that $4^2\cdot y=144$, so $y=\boxed{9}$.

16. Note that $49^{27x}=(7^2)^{27x}=7^{54x}$. As a result, $\dfrac{49^{27x}}{7^{9x}}=7^{45x}$. Since this is equal to $49=7^2$, we have that $x=\boxed{\dfrac{2}{45}}$.

17. We know that all squares of real numbers are nonnegative. Thus, if two squares add to 0, both of them must be zero. We must therefore have $2x+3y=5$ and $x-2y=-7$. Tripling the second equation and adding it to twice the first equation gives us $7x=-11 \implies x=\boxed{-\frac{11}{7}}$.

18. The resulting figure has one circular face, of surface area πr^2, one lateral surface of a cylinder, with surface area $2\pi rh$, and one-half of a sphere, which has surface area $\frac{1}{2}(4\pi r^2)=2\pi r^2$. Thus, the total surface area is $3\pi r^2+2\pi rh=3\pi(5^2)+2\pi(5)(10)=\boxed{175\pi}$.

19. A number has exactly 3 divisors if and only if it is the square of a prime number. The third smallest number of this form is the square of the third smallest prime, 5, so the answer is $5^2=\boxed{25}$.

20. It can be easily verified that the statement holds true when $n=3,4,$ or 5. For $n>5$, it can easily be seen that the longest diagonal is longer than the shortest. Thus, our answers are $\boxed{3,4,\text{ and }5}$.

21. If $2x+5y=100$, since $2x$ and 100 are both even, $5y$ must be even too, so y must be even. Since $x\geq 6$, $5y\leq 100-2x\leq 88$, so $y\leq 17$. From here, we can list all the solutions: $(x,y)=(35,6),(30,8),(25,10),(20,12),(15,14),(10,16)$. There are $\boxed{6}$ solutions in this list.

22. In the binomial expansion of $\left(x+\frac{1}{x}\right)^{2011}$, all the terms are of the form $cx^m\left(\frac{1}{x}\right)^n$, with $m+n=2011$. In order to attain the term x^2, we must have $m-n=2$. Solving these equations gives $(m,n)=(\frac{2013}{2},\frac{2009}{2})$. But since m and n must be integers, the x^2 term must have coefficient $\boxed{0}$.

23. The number 9^n ends with 1 if n is even, and ends with 9 if n is odd. Since $8^{7^{6^{5^{4^{3^{2^{1}}}}}}}$ is even, the last digit of $9^{8^{7^{6^{5^{4^{3^{2^{1}}}}}}}}$ is $\boxed{1}$.

24. First of all, we must consider that 2012 is a leap year so we must consider the extra day (hence 2012 will have 366 days instead of 365). We must also consider that October 16, 2011, the day of the contest, is a Sunday. One year from now, which is 366 days later, October 16, 2012 will be in 52 weeks and 2 days, so October 16 will be a Tuesday. Therefore, October 17 will be a $\boxed{\text{Wednesday}}$.

25. The expression on the left hand side will be equal to 1 only when the exponent is equal to 0 or if $x+1$ is equal to -1 or 1. The exponent will be equal to 0 when $x=-1$ or -2, and the base is equal to 1 when $x=-2$ or when $x=0$. We verify that $x=0$ and $x=-2$ satisfy the equation, while $x=-1$ yields an expression of the form 0^0, which is undefined, so the only x satisfying the equation are 0 and -2, so our answer is $\boxed{2}$.

26. Subtract the second equation from the first to get $3x^2-8x=35 \implies 3x^2-8x-35=0$. This factors as $(3x+7)(x-5)=0 \implies x=5$ or $x=-\frac{7}{3}$. However, only one of these is a solution to the first equation, namely $x=\boxed{5}$.

27. After drawing the figure, we find that half the chord (which has a length of 6), the radius of the circle, and the line from O to AB form a right triangle with legs 3 and 6. Applying

the Pythagorean Theorem, we find that the hypotenuse is $3\sqrt{5}$, which is also the radius. The area is therefore $\pi \cdot \text{radius}^2 = \pi \left(3\sqrt{5}\right)^2 = \boxed{45\pi}$.

28. If n is a number, it satisfies the property if and only if each of the numbers divides $n+2$. Thus, we just need to find the least common multiple of 4, 5, and 6 and subtract 2. The lowest common multiple is 60, so our answer is $\boxed{58}$

29. If the first roll is a 1, we have six choices for the second roll. If it is a 2, that gives us three choices for the second roll (2, 4, 6). If it is a 3, then there are only two choices (3, 6), and for 4, 5, and 6, there is only one choice (itself). That gives us a probability of $\frac{6+3+2+1+1+1}{36} = \boxed{\frac{7}{18}}$.

30. By the factor theorem, $x^3 - 6x^2 + 5x + 1$ factors as $(x-r)(x-s)(x-t)$. Thus, $(2-r)(2-s)(2-t) = 2^3 - 6 \cdot 2^2 + 5 \cdot 2 + 1 = 8 - 24 + 10 + 1 = \boxed{-5}$.

31. $((3!)!)! = (6!)! = 720!$; the number of zeroes at the end of 720 is $\lfloor \frac{720}{5} \rfloor + \lfloor \frac{720}{25} \rfloor + \lfloor \frac{720}{125} \rfloor + \lfloor \frac{720}{625} \rfloor = 144 + 28 + 5 + 1 = \boxed{178}$.

32. Let $x = BD$ and $y = DC$. Clearly, $x + y = 5$. Also, by the angle bisector theorem, $\frac{x}{y} = \frac{4}{6} = \frac{2}{3}$, so $3x = 2y = 2(5-x)$, so $5x = 10$, so $x = BD = \boxed{2}$.

33. $1^2 - 2^2 + 3^2 - 4^2 + \cdots - 50^2 + 51^2 = 1 + (3+2)(3-2) + (5+4)(5-4) + \cdots + (51+50)(51-50)$. (Note: $a^2 - b^2 = (a+b)(a-b)$.) This is equal to $1 + 5 + 9 + \cdots + 101 = 13 \cdot 102 = \boxed{1326}$.

34. Using the Pythagorean Theorem on the diagonal of the rectangle gives us that it has length $\sqrt{10^2 + 24^2} = 26$. Since the diagonal is the hypotenuse of a right triangle inscribed in a circle, it is also a diameter. Thus the radius is 13 and its area is $13^2\pi = \boxed{169\pi}$.

35. Note that the line segments A_1B_1, B_1C_1, C_1A_1 divides triangle ABC into four identical smaller triangles, and similarly A_2B_2, B_2C_2, C_2A_2 divides triangle $A_1B_1C_1$ into four even smaller identical triangles. Therefore, the area of triangle ABC is four times that of triangle $A_1B_1C_1$, which in turn has area four times that of $A_2B_2C_2$. Hence, the ratio of the area of triangle $A_2B_2C_2$ to the area of triangle ABC is $\boxed{\frac{1}{16}}$.

36. Since Archimedes' position in the line is fixed, it is enough to find the number of ways to arrange Bernoulli, Cauchy, Descartes, and Euler in a line. Because Descartes and Euler must stand next to each other, we temporarily view them as a single unit. The number of ways to arrange Bernoulli, Cauchy, and the "Descartes-Euler unit" is $3 \times 2 \times 1 = 6$, and the number of ways to arrange Descartes and Euler in the "unit" is 2, so our answer is $6 \times 2 = \boxed{12}$.

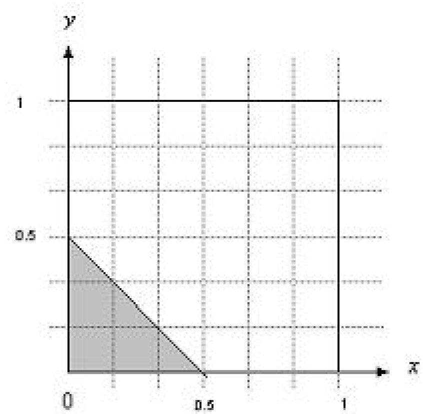
37. Let y be the length of the side parallel to the river, and let x be the lengths of the other two sides. We must have $2x + y = 200$. We seek to maximize the area, xy, that is enclosed by the fence and the river. Rearranging, we get $y = 200 - 2x$, so the area is $xy = x \cdot (200 - 2x) = 200x - 2x^2$, which is a quadratic in x. The vertex, or maximum (because the coefficient of x^2 is negative), occurs at $x = \frac{-200}{(2 \cdot (-2))} = 50$. When $x = 50, y = 200 - 2 \cdot 50 = 100$. The area is then $xy = 50 \cdot 100 = \boxed{5000}$.

38. The expression factors to $(x+4)(x-2)$, which can only be a prime number if one of the two expressions is equal to 1 or -1 and the other one is a prime number or a negative number whose absolute value is prime. The only values of x that work are $\boxed{-5}$ and $\boxed{3}$.

39. Call the roots of $x^2+4x-2=0$ a and b. From Vieta's formulas, we know that $ab=-2$ and $a+b=-4$. Write the desired quadratic as x^2-qx+r, for some q and r. Again by Vieta's forumlas, we have that $r=(ab)^2$ and $q=a^2+b^2$. r is simply $(-2)^2=4$. To find q, we note that $(a+b)^2=a^2+2ab+b^2$. Rearranging, we have that $a^2+b^2=(a+b)^2-2ab=(-4)^2-2(-2)=20$. The quadratic equation we seek is therefore $\boxed{x^2-20x+4=0}$.

40. There are 50 cards from which Steven can draw. The cards he can draw to keep from going above 21 are each of the four aces, twos, threes, fours, and fives, the three remaining sixes, the three remaining sevens, and the four eights. There are 30 such cards in total, so the probability is $\frac{30}{50}=\boxed{\frac{3}{5}}$.

41. Consider the attached diagram. If we choose x,y randomly in this 1×1 square, the probability that their sum is less than $\frac{1}{2}$ is the probability that (x,y) lies in the shaded region. The area of the shaded region is $\frac{1}{8}$, while the area of the whole region is 1, so our answer is $\boxed{\frac{1}{8}}$.



42. Let a,t,r be the percentage of the apples that Applejack, Twilight Sparkle, and Rainbow Dash can pick in one day, respectively. We are given that $3(a+t)=4(a+r)=6(t+r)=1$, so $a+t=\frac{1}{3}$, $a+r=\frac{1}{4}$, and $t+r=\frac{1}{6}$. Adding these up and dividing by 2 gives $a+r+t=\frac{3}{8}$, so it will take $\frac{1}{a+r+t}=\boxed{\frac{8}{3}}$ days for them to pick all the apples.

43. Let us consider $n \mod 3$. When $n\equiv 0 \mod 3$, $n^2+2n+3\equiv 0+0+3\equiv 0 \mod 3$. When $n\equiv 1 \mod 3$, $n^2+2n+3\equiv 1+2+3\equiv 0 \mod 3$. Finally, when $n\equiv 2 \mod 3$, $n^2+2n+3\equiv 4+4+3\equiv 2 \mod 3$. Thus, $3|n^2+2n+1$ exactly when n is congruent to 0 or 1 mod 3. There are 670 numbers that are $0 \mod 3$, namely $3\cdot 1, 3\cdot 2, \ldots, 3\cdot 670=2010$. There are 671 numbers that are $1 \mod 3$, namely $3\cdot 0+1, 3\cdot 1+1, \ldots, 3\cdot 670+1=2011$. Thus there are $670+671=\boxed{1341}$ numbers that satisfy the restrictions.

44. $11_2 + 11_3 + 11_4 + \cdots + 11_100 = (1 \times 2 + 1) + (1 \times 3 + 1) + (1 \times 4 + 1) + \cdots + (1 \times 100 + 1) = 3 + 4 + 5 + \cdots + 101$. This is the same as the sum of positive integers up to 101 without 1 and 2. The sum of positive integers from 1 to n can be expressed as $\frac{n(n+1)}{2}$. The sum is therefore $\frac{101(102)}{2} - (1+2) = \boxed{5148}$.

45. Consider the top left hand corner square of an $n \times n$ square. Notice that when $n = 1$, there are 6^2 choices for the top left hand corner square, when $n = 2$, there are 5^2 choices, and, in general, for an $k \times k$ square there are $(7-k)^2$ choices for the top left hand corner square. Thus, since k ranges from 1 to 6, the total number of squares is $1^2+2^2+3^2+4^2+5^2+6^2 = \boxed{91}$

46. Consider the 3 balls Bob picks. The only way one of your balls will not be among Bob's balls is if you pick the two he did not pick. This only happens with probability $\frac{1}{\binom{5}{2}} = \frac{1}{10}$, so the probability one of your balls will be among Bob's chosen ones is $1 - \frac{1}{10} = \boxed{\frac{9}{10}}$.

47. If $x - \frac{1}{x} = 3$, then $\left(x - \frac{1}{x}\right)^2 = 9$, or $x^2 + \frac{1}{x^2} - 2 = 9$, so $x^2 + \frac{1}{x^2} = \boxed{11}$.

48. The ratio of the areas of two similar polygons is the square of the ratio of similitude between the two polygons, so we need to find $\left(\frac{B_1B_2}{A_1A_2}\right)^2$. If we let M be the midpoint of B_1B_2, we see that A_2B_2M is a 30-60-90 triangle, so $\frac{B_2B_1}{A_3A_2} = \frac{B_2M}{B_3A_2} = \frac{\sqrt{3}}{2}$, so the desired ratio is $\left(\frac{\sqrt{3}}{2}\right)^2 = \boxed{\frac{3}{4}}$.

49. First, notice that $\frac{x^5-1}{x-1} = x^4+x^3+x^2+x+1$. Letting $x = 10^{20}$ gives us that the subtrahend is $(10^{80}+10^{60}+10^{40}+10^{20}+1)$. Next, we realize that $10^{100}-1 = 999\ldots999$, or a string of 100 9's, which has a sum of 900. Since we are subtracting the number $(10^{80}+10^{60}+10^{40}+10^{20}+1)$, which is composed of a total of 5 1's and many 0's, and since no borrowing is done, the answer is just $900 - 5 = \boxed{895}$.

50. The area of a polygon with 20,000 sides can be approximated by the area of a circle. The polygon has a diagonal of length 20, which corresponds to a diameter of 20 in the approximating circle. A circle with a diameter of 20 has a radius of 10 for an area of $\pi \cdot 10^2 = 100\pi$, which is approximately $\boxed{314}$.