Joe Holbrook Memorial Math Competition

8th Grade

October 9th, 2016

General Rules

- You will have **90 minutes** to solve **50** questions. Your score is the number of correct answers.
- Only answers recorded on the answer sheet will be graded.
- This is an individual test. Anyone caught communicating with another student will be removed from the exam.
- Scores will be posted on the website. Please do not forget your ID number, as that will be the sole means of identification for the scores.
- You may **not** use the following aids:
  - Calculator or other computing device
  - Compass
  - Protractor
  - Ruler or straightedge

  In addition, you must use the scrap paper supplied by the proctors.

Other Notes

- Write legibly. If the graders cannot read your answer, you will be given no credit for that question.
- Fractions should be written in **lowest terms**. Please convert all mixed numbers into **improper fractions**.
- For constants such as e or π, do not approximate your answer: for example, if the answer to a question is 7π, then you should **not** write 22 or 21.99.
- You do not need to write units in your answers.
- Rationalize all denominators. In addition, numbers within a square root must be squarefree, e.g. $\sqrt{63}$ should be written as $3\sqrt{7}$.
- Ties will be broken by the number of correct responses to questions 41 through 50. Further ties will be broken by the number of correct responses in the last five questions.

1. Compute $2 + (0 - (1 \cdot 6 \cdot (2^{0 \cdot (\frac{1}{6})})))$.
2. Two regular polygons have angles of 135 degrees and 108 degrees. Find the sum of the number of sides on both polygons.
3. When Kelvin the Frog was a tadpole, he took the JHMMC. His score improved by the same number of questions every year. If he got 31 questions right in 4th grade and 43 questions right in 8th grade, how many did he get right in 7th grade?
4. The number of lilypads in Kelvin the Frog's pond doubles every day. If there were 48 lilypads on Saturday, on what day of the week did he have an odd number of lilypads?
5. If $f(x) = x + 1$, $g(x) = \lfloor x \rfloor$, and $h(x) = g(f(x))$, compute $h(\pi^2)$.
6. What is the sum of the roots of $x^2 - 2x + 3$?
7. Suppose 3 flips are worth 5 flops and 9 flops are worth 14 flaps. How many flaps are equal to 54 flips?
8. Letter blocks with the letters "A", "B", and "C" are in a bag. If you take out a letter at a time, what is the probability that they come out in the order spelling "BCA"?
9. Arthur and Sunny are running a 100 meter race against each other. For the first 5 seconds, Arthur runs at 8 m/s. Tired, he slows down to 3 m/s for the rest of the race. Meanwhile, Sunny opened the first 8 seconds of the race running only at 4 m/s. If he wants to at least tie Arthur, what is the minimum speed he must run at for the rest of the race? Express answer as a common fraction.
10. In how many distinct words (strings of letters) can be formed by permuting all of the letters of $JHMMC$?
11. Compute the units digit of $2^{2016} + 3^{2016}$.
12. The area of a square is 25. The area of an equilateral triangle is $9\sqrt{3}$. What is the difference in side lengths?
13. What is the least positive integer value of n such that $|n! - n^3| > 2016$?
14. Andrew, Ben, and Caleb want to split their n coins. They agreed that Andrew should get $\frac{1}{2}$ of the coins, Ben should get $\frac{1}{3}$, and Caleb should get $\frac{1}{9}$, with the rest going to charity. However, their coins won't split evenly! Their friend Dennis comes and loans them a coin. Now, they split the coins as decided before, and one coin is left. Find n.
15. What is the largest integer value for x such that $\dfrac{x}{x+2} < \dfrac{61}{64}$?
16. Jake can make 12 mini cheesecakes in half an hour. If he works with his brother, Zach, they can make 84 cheesecakes in two hours. How many can Zach make in 4 hours?
17. Jake is once again making mini cheesecakes. They are cylinders with radius $\frac{3}{2}$ and height 1. If he feels generous one day and wants to increase the volume by 44% and keep the same height, what should he make the radius of the cakes?
18. How many numbers less than 2016 have exactly 3 factors?
19. Two circles centered at points A and B, are tangent to each other at a point C. A common external tangent to circles A and B is tangent at X and Y, respectively. Find $\angle XCY$.
20. Everyone in DrizzleLand has a 1 in 150 chance of having the muggy virus. The test for this virus has a 96% of giving the correct result for any patient, whether or not he or she has the virus. If Zach tests positive, what is the probability that he actually has the muggy virus?
21. How many ways are there to pick 3 (not necessarily distinct) non-negative integers in order such that the sum of the first two is equal to 10 minus the third integer?
22. Given a circle with a radius of 6, what is the area of the largest square that can be inscribed in the circle?
23. Triangles ABC and XYZ are similar. It is known that $AB = 15$, $BC = 10$, $XY = 3$, and $\angle ABC = 30°$. Find the area of triangle XYZ.

24. Simplify $\sqrt{9-\sqrt{17}}+\sqrt{9+\sqrt{17}}$.

25. Consider an 8 digit number using each of the digits $1-8$ exactly once, with the following conditions:

    The greatest common factor of the first two digits is greater than 1. The next two digits are both prime numbers. The next two digits are both perfect squares. The last two digits are both triangular numbers.

    What is the smallest number that satisfies the given conditions?

26. What is $1^2-3^2+5^2-7^2+\cdots+65^2-67^2$?

27. Jonathan purchased a bulk order of Jake's mini cheesecakes, and invites his friends Antonia, Jordan, and Amber. However, they realize that after splitting the cakes evenly, there is one cake left. To solve the problem, they invite Victor. But after splitting evenly again, there is still one cake left! Flabbergasted, they invite David and Youjung as well. But luck has it that after splitting the cakes one more time, there are now *two* cakes left.

    If Jonathan purchased more than 150 cakes, what is the minimum number of cakes he could have bought?

28. Find distance between intersections of the graphs of $y=2x+3$ and $y=\frac{5}{7}x^2-\frac{26}{7}x+8$.

29. In DrizzleLand, there are twelve major cities connected by a network of roads such that the roads only intersect at the major cities. If the four largest cities each have 7 roads leading out of them and the eight smallest cities each have 5 roads leading out of them, how many total roads are in DrizzleLand?

30. Find the remainder when $1+2+4+8+\cdots+2^{2015}$ is divided by 7.

31. In a standard 8×8 chessboard, how many rectangles are there?

32. Compute the sum $\frac{7}{\sqrt{1}+\sqrt{2}}+\frac{7}{\sqrt{2}+\sqrt{3}}+\frac{7}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{7}{\sqrt{48}+\sqrt{49}}$.

33. Jake is selling cheesecakes in boxes of 5 and 12. What is the greatest number of cheesecakes that you are unable to obtain by purchasing whole boxes?

34. Arthur the Aardvark stands before 5 ant nests. Arthur is a highly trained sharpshooter, and thus, the probability that he hits a nest with a single shot of his tongue is $\frac{4}{5}$. He shoots at each of the nests. What is the probability that an even number of nests are hit?

35. How many 6-digit numbers start with 7 and have exactly 2 identical digits?

36. Point O is selected inside a rectangle $ABCD$ such that $OA=4$ and $OC=11$. Find OB^2+OD^2.

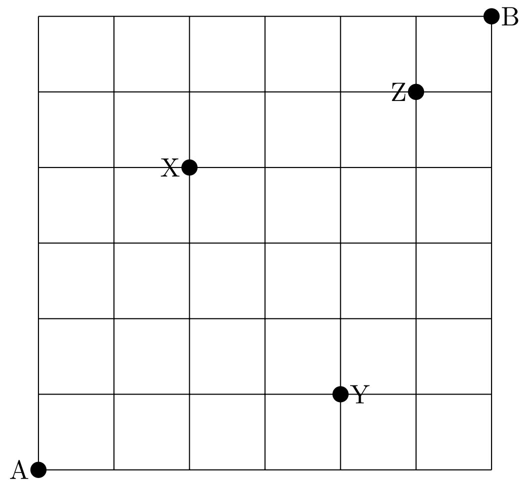
37. Let a, b, c, d be distinct digits. A real number is called "cool" if it can be represented as a repeating decimal of the form $0.\overline{abcd}$. Find the sum of all "cool" numbers.

38. Find the number of integers between 1 and 350 inclusive whose sum of distinct prime divisors is 18 (For example, the integer 12 would have a sum of $2+3=5$)?

39. Compute the infinite sum $\frac{1}{2}+\frac{3}{4}+\frac{5}{8}+\frac{7}{16}+\ldots$.

40. Determine the number of six-digit integers $\overline{810abc}$ such that each of the numbers $\overline{810abc}, \overline{810cab}, \overline{810bca}$ is divisible by 27.

41. Suppose we have the following grid. How many ways are there to get from A to B such that you don't pass through the points X,Y, and Z and such that you can only move right or up.



42. A regular hexagon $A_1B_1C_1D_1E_1F_1$ has side length 4. The midpoints of adjacent sides are connected, to form a new hexagon $A_2B_2C_2D_2E_2F_2$. The same is done to hexagon $A_2B_2C_2D_2E_2F_2$ to form hexagon $A_3B_3C_3D_3E_3F_3$. This is done infinitely to form an infinite amount of hexagons. What is the combined area of the regions that are in the interior of an odd amount of hexagons?

43. Let x be the length of the shortest path between the points $(4, 9)$ and $(12, 4)$ that touches both the x-axis and the y-axis at least once. Find x^2.

44. How many of the first 100 positive integers can be expressed in the form $\lfloor 2x \rfloor + \lfloor 3x \rfloor + \lfloor 4x \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, and x is a positive real number?

45. Let all k positive divisors of 2016^2 be $d_1, d_2, ...d_k$. What is the sum $\frac{1}{d_1 + 2016} + \frac{1}{d_2 + 2016} + ... + \frac{1}{d_k + 2016}$?

46. The graph of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is rotated counterclockwise about the origin by an angle of $\pi/4$ radians. The equation of the resulting graph can be written in the form $ax^2 + bxy + cy^2 = 32$. Compute $a + b + c$.

47. If $x^2 + 17 - 16y = -y^2 + 12x + 13$, find the maximum value of $x + y$.

48. In orthodiagonal quadrilateral $ABCD$, $AC = 12$ and $BD = 18$. Points M and N are the midpoints of AB and CD, respectively. If the projection from point M onto CD intersects CD at P and $NP = 6$, find MP.

49. In triangle $\triangle ABC$, the angle bisectors of angles A, B, C concur at the incenter I. Given that $AB = 13, BC = 14, CA = 15$, find the square of the area of a triangle with side-lengths AI, BI, CI.

50. Let $[n]$ denote the number $111 \ldots 111$ with n digits and $\{m\}$ denote the number $101010 \ldots 101010$ with $2m$ digits. What is the biggest n such that $[n] | \{1008\}$? ($a|b$ denotes "a divides b" for some integers a and b).