Joe Holbrook Memorial Math Competition

7th Grade Solutions

October 9th, 2016

1. Since half an hour is 30 minutes, Kelvin the Frog can listen to his favorite song $\frac{30}{6} = \boxed{5}$ times.

2. $2 + (0 - (1 \cdot 6(2^{0 \cdot \frac{1}{6}}))) = 2 + (0 - (1 \cdot 6(2^0))) = 2 + (0 - (1 \cdot 6)) = 2 - 6 = \boxed{-4}$.

3. Alex the Kat needs to write x questions such that $25 + x = 40$, so $x = 40 - 25 = \boxed{15}$.

4. Zack made half of eight shots, meaning he made $0.5 \cdot 8 = 4$ shots. Each shot was worth 3 points, so he made a total of $4 \cdot 3 = \boxed{12}$ points.

5. We have $4^4 = 256$, and $2^2 = 4$, so the answer is $\frac{256}{4} = \boxed{64}$. Alternatively, note that $4 = 2^2$, so $\frac{4^4}{2^2} = \frac{4^4}{4} = 4^3 = \boxed{64}$.

6. By the formula that says the interior angle of a regular n-sided polygon can be found by $\frac{180 \cdot (n-2)}{n}$, we can see that an 8-sided polygon (octagon) has angles of 135 degrees and a 5-sided polygon (pentagon) has angles of 108 degrees. Therefore, 8+5=$\boxed{13}$.

7. The prime factorization of 2016 is $2^5 \cdot 3^2 \cdot 7$. The prime factorization of 2772 is $2^2 \cdot 3^2 \cdot 7 \cdot 11$. The greatest common factor can be found by identifying the least exponent of each prime factor: $2^2 \cdot 3^2 \cdot 7 = \boxed{252}$.

8. $2^{11} = 2048$, and $2^{10} = 1024$. Therefore, the smallest n satisfying the equation is $\boxed{11}$.

9. Four years pass between Kelvin's 4th and 8th grades, which means that his scores improved 4 times. During this period, his score increased by $43 - 31 = 12$ points. Since he increased by an equal amount every year, we divide the total increase by the total time to get $\frac{12}{4} = 3$. This means that in 7th grade he scored 3 fewer points than in 8th grade, or $43 - 40 = \boxed{40}$.

10. Since doubling any number gives us an even number, we have to work backwards. On Friday, Kelvin had $\frac{48}{2} = 24$ lilypads. On Thursday, he had $\frac{24}{2} = 12$ lilypads. On Wednesday, he had $\frac{12}{2} = 6$ lilypads. On Tuesday, he had $\frac{6}{2} = 3$ lilypads, making our day $\boxed{\text{Tuesday}}$.

11. Let the number of points Marvin scored be M and the number of points June scored be J. We are given that $M + J = 34$, and since Marvin won, $M = 21$. Therefore, $J = 34 - M = 34 - 21 = 13$. June scored $\boxed{13}$ points.

12. Every 7 days after a Friday is also a Friday. Therefore, if you divide 2016 by 7, the quotient is irrelevant, and the remainder tells you how many days of the week you must count before reaching the correct day of the week. However, since 2016 is perfectly divisible by 7, the remainder is 0, so the day of the week will be the same. Therefore, the 2016^{th} day after January 1st, 2016 will be on a $\boxed{\text{Friday}}$.

13. The value of $f(\pi^2)$ is simply π^2+1, and $g(\pi^2+1) = \lfloor \pi^2+1 \rfloor = 10$, as π^2 lies between 9 and 10. Thus, $h(\pi^2) = \boxed{10}$.

14. There are 3 different choices for buying milk, 2 different choices for buying eggs, and 4 different choices for buying butter, so the total number of ways to buy one of each is equal to $3 * 2 * 4 = \boxed{24}$.

15. Using Vieta's formula, the sum is $\dfrac{-(-2)}{1} = \boxed{2}$.

16. 54 flips $= 18 \cdot 5 = 90$ flops. 90 flops $= 10 \cdot 14 = \boxed{140}$ flaps.

17. There is a probability of $\dfrac{1}{3}$ of pulling out the letter B first; then a probability of $\dfrac{1}{2}$ of pulling out C; the letter A then has a $\dfrac{1}{1}$ chance of being selected. Multiplying the fractions together gives a total probability of $\boxed{\dfrac{1}{6}}$.

18. Arthur ran 40 meters in the first 5 seconds. He only has to run for $\dfrac{100-40}{3} = \dfrac{60}{3} = 20$ more seconds. Sunny ran for 32 meters in the first 8 seconds. That means that in $25-8=17$ seconds, he must run 68 meters, which is an average speed of $\dfrac{68}{17} = \boxed{4}$ m/s.

19. The number of permutations disregarding the repeated alphabat is $5! = 120$. However, the letter M is repeated twice, thus the number should be divided by $2!$, yielding $\boxed{60}$ as the answer.

20. Note that 2^4 has a units digit of 6. Since $2^{2016} = (2^4)^{504} = 6^{504}$, and every power of 6 ends in 6, we know 2^{2016} has a units digit of 6. Also note that $3^4 = 81$ has a units digit of 1. Since $3^{2016} = (3^4)^{504} = 81^{504}$, we know 3^{2016} has a units digit of 1. Our answer is therefore $1+6 = \boxed{7}$.

21. Using the common area formulas for both figures: The square's side s is the solution to $s^2 = 25$, and is therefore 5. For the equilateral triangle, $\dfrac{s^2 \cdot \sqrt{3}}{4} = 9\sqrt{3}$, so the side length is 6. The difference between the two is $6-5 = \boxed{1}$.

22. The largest number can be constructed by selecting 9, then 6 (since 7 is also odd, and 8 is too close to 9), then 9, then 6, then 9 again. Similarly, the smallest legal number is 14141. Thus the sum is $\boxed{111110}$.

23. We note that $\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{9} = \dfrac{17}{18}$. Therefore, there will be $\dfrac{1}{18}$ of the coins left after the division. Since there was 1 coin left after Dennis lent the group a coin, there must have been 18 coins in total after the loan. Since we're looking for the original number of coins, we subtract 1 and get $\boxed{17}$.

24. The ratio of the areas is $\dfrac{360}{40} = 9$, hence the ratio of the sides will be $\sqrt{9} = 3$. The length h of the larger hypotenuse will satisfy $\dfrac{h}{15} = 3$, and we find $h = \boxed{45}$.

25. Since $\dfrac{x}{x+2} < \dfrac{61}{64}$, multiplying both sides of the inequality by $64(x+2)$ yields $64x < 61(x+2)$, which can be simplified to $3x < 122$, then the largest integer value for x would be $\boxed{40}$.

26. Jake makes 12 cakes per half hour, or 24 per hour. Together, the brothers make 84 cakes in two hours, or 42 per hour. Jake makes 24 of these, leaving Zach to make $42-24 = 18$ per hour. In four hours, he makes $18 \cdot 4 = \boxed{72}$.

27. The volume of a cylinder is $\pi r^2 h$, where r is the radius and h is the height. If the new cheesecake is 44% greater in volume, that means that its volume is 144% of the original cheesecake, or the ratio of the two volumes is $\frac{144}{100} = \frac{36}{25}$. If we call the new radius r, then we have that $\frac{\pi r^2 h}{\pi \frac{3}{2}^2 h} = \frac{r^2}{\frac{3}{2}^2}$ also equals this value. Taking the square root of both sides, we have $\frac{r}{\frac{3}{2}} = \frac{6}{5}$, or $r = \boxed{\frac{9}{5}}$.

28. Numbers with an odd number of factors are perfect squares, since the number's square root only counts as a factor once. Numbers with exactly 3 factors must be perfect squares of prime numbers, because in that case the only factors would be 1, itself, and its square root. The square root of 2016 is slightly less than 45 ($45^2 = 2025$), so we must consider the squares of all prime numbers less than 44. These numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, and 43, for a total of $\boxed{14}$ numbers.

29. If we add the two equations, we get that $2 \cdot (x+y)^4 = 1 + 3 = 4$, so $(x+y)^4 = \frac{4}{2} = 2$, so $(x+y) = \pm 2^{\frac{1}{4}}$. Since we are asking for the absolute value, take the positive answer of $\boxed{2^{\frac{1}{4}}}$.

30. The probability of the test being positive is $\frac{1}{150} \cdot \frac{96}{100} + \frac{149}{150} \cdot \frac{4}{100} = \frac{692}{15000}$, The probability of a person having the virus and testing positive is $\frac{1}{150} \cdot \frac{96}{100} = \frac{96}{15000}$. Thus, the probability that Zach has the muggy virus given that he tested positive is $\frac{\frac{96}{15000}}{\frac{692}{15000}} = \boxed{\frac{24}{173}}$.

31. The degree measure of every interior angle of a regular pentagon is $108°$. Then the region along which David can travel is simply a circle of radius 3 with a $108°$ sector cut out. Since $360 - 108 = 252$, the area of this will be $\frac{252}{360} \cdot 9\pi = \boxed{\frac{63\pi}{10}}$.

32. The units digit obviously must be even. Suppose it is 4. Then the tens digit must be 6 for the number to be a multiple of 4. That gives us 2 ways to arrange the 5 and 7 among the other two digits, so 2 total arrangements so far. Now consider when the units digit is 6. The tens digit must be odd. If it is 5, there are 2 ways to arrange the remaining digits, and if it is 7 there are 2 ways to arrange the remaining digits. That gives us 2+2+2=6 total ways. There are 4!=24 total ways to permute the digits, so the answer is $\frac{6}{24} = \boxed{\frac{1}{4}}$.

33. The largest square would have the diameter of the circle as one of its diagonals (it would have length 12), then the side lengths of the square would be $\frac{12}{\sqrt{2}}$, and squaring that yields $\boxed{72}$.

34. From the similarity, $\angle XYZ = 30°$, and $\frac{YZ}{BC} = \frac{XY}{AB} \implies YZ = 2$. Using the sine area formula in XYZ, we see that $[XYZ] = \frac{1}{2} \cdot \sin 30° \cdot 2 \cdot 3 = \boxed{\frac{3}{2}}$.

35. The best way to approach this problem is to consider all of the ways to satisfy the given conditions with pairs of numbers. Notice that there are only two perfect squares from 1-8: 1 and 4. This leaves two remaining triangular numbers, 3 and 6, since 1 cannot be used again.

    This leaves us with the digits 2, 5, 7, and 8. Of these, 2, 5, and 7 are prime. This means that 8 must be one of the first two digits. The other number that must be one of the first two digits must share a factor with 8 other than 1. The only number that satisfies these conditions is 2. Therefore, 5 and 7 are the next two digits, which checks out because they are both prime.

    The four pairs of numbers are (8 2), (5 7), (1 4), and (6 3), respectively. To get the smallest possible 8-digit number, simply list the smaller digit in each pair first. This gives a result of $\boxed{28571436}$.

36. Sort the numbers in groups of two, $(1^2-3^2)+(5^2-7^2)+\cdots+(65^2-67^2)$, then writing each as a difference of squares yields, $(1+3)(1-3)+(5+7)(5-7)+\cdots+(65+67)(65-67)$. This can then be rewritten as $-2(1+3+5+7+...+65+67)$. The sum of the arithmetic series $1+3+\cdots+67$ is 1156, so $-2(1156) = \boxed{-2312}$.

37. Let $BD = x$. Since ABD is a 45-45-90 triangle (and hence isosceles), AB also equals x. Using the side ratios for $30-60-90$ triangles, $x\cdot\sqrt{3} = BC = BD + DC = x + 2 - \sqrt{3}$. Solving for x results in $x = \boxed{\frac{\sqrt{3}-1}{2}}$.

38. We know that the prime factorization of 2016 is $2^5 \cdot 3^2 \cdot 7^1$. To list the sum of all of the divisors of 2016 we know that we can look at the following product

$$\left(2^0+2^1+2^2+2^3+2^4+2^5\right)\left(3^0+3^1+3^2\right)\left(7^0+7^1\right)$$

All of the divisors of 2016 are one of the elements in the expansion of the product above. Note that all odd factors in the sum come from the product of 2^0 and some combination of the other factors. Thus, to find the sum of the even factors, we have to look at the following product,

$$\left(2^1+2^2+2^3+2^4+2^5\right)\left(3^0+3^1+3^2\right)\left(7^0+7^1\right)$$

This evaluates to $\boxed{6448}$.

39. First we notice that the first three terms sum to 7. For every three term sequence after, taking modulo 7, we can see that every sequence sums to $1+2+4$. Thus, every three term segment leaves no remainder when divided by 7. Since the entire sequence has 2016 terms which is a multiple of 3, a whole number of three-term sequences are contained within the series and the sum leaves a remainder of $\boxed{0}$ when divided by 7.

40. Factoring out the 7 and rationalizing the denominators yields

$$7\left(\frac{\sqrt{2}-\sqrt{1}}{2-1} + \frac{\sqrt{3}-\sqrt{2}}{3-2} + \cdots + \frac{\sqrt{49}-\sqrt{48}}{49-48}\right) = 7(\sqrt{49}-\sqrt{1}) = \boxed{42}.$$

41. We use casework on the number of dogs that die (0, 2, or 4). The probability that 0 dogs die is $\left(\frac{1}{5}\right)^5$. The probability that 2 dogs die is $\binom{5}{2}\cdot\left(\frac{4}{5}\right)^2\cdot\left(\frac{1}{5}\right)^3$. Finally the probability that 4 dogs die is $\binom{5}{4}\cdot\left(\frac{4}{5}\right)^4\cdot\left(\frac{1}{5}\right)^1$. Summing up yields a probability of $\boxed{\frac{1441}{3125}}$.

42. We are not given the dimensions of rectangle $ABCD$, so this should suggest that the exact dimensions are not important, hinting at a general solution that applies for every rectangle. Then, let us consider an arbitrary point O in a rectangle $ABCD$. Construct EF through O such that E lies on AB, F lies on CD, and $EF \parallel AD$. Similarly, construct GH through O such that G lies on AD, H lies on BC, and $GH \parallel AB$. Notice that $AC = BH$, $GD = CH$, $AE = DF$, and $EB = FC$. Then we can write expressions for the distance between O and each of the rectangle's vertices in terms of these lengths.

$$OA^2 = AG^2 + AE^2$$
$$OB^2 = EB^2 + BH^2$$
$$OC^2 = FC^2 + CH^2$$
$$OD^2 = FD^2 + GD^2$$

From this, we realize that $OA^2 + OC^2 = OB^2 + OD^2$ for any generic rectangle and any arbitrary O. Then, we can simply apply this fact back to the original problem, where we are given both OA and OC.

$$OB^2 + OD^2 = OA^2 + OC^2 = 4^2 + 11^2 = \boxed{137}.$$

43. Since a, b, c, d are distinct digits, there are $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ "cool" numbers. Now, the repeating decimal $0.abcd$ can also be written as $\dfrac{1000a + 100b + 10c + d}{9999}$. Notice that every digit m has a unique digit $n \neq m$ such than $m + n = 9$. Therefore, for all "cool" numbers $P = 0.abcd$, there is another "cool" number $Q = 0.efgh$ such that $P + Q = \dfrac{1000a + 100b + 10c + d}{9999} + \dfrac{1000e + 100f + 10g + h}{9999} = \dfrac{1000(a+e) + 100(b+f) + 10(c+g) + (d+h)}{9999} = \dfrac{9999}{9999} = 1.$ Since there are 5040 "cool" numbers, there are $\dfrac{5040}{2} = 2520$ such pairs that add up to 1, so our sum is $\boxed{2520}$.

44. Because 810000 is divisible by 27, the three numbers are divisible by 27 if and only if $\overline{abc}, \overline{cab}$, and $\overline{bca}$ are all divisible by 27. We consider when one of the three is divisible by 27 by expressing one in terms of another. Since $\overline{bca} = 10 \cdot \overline{abc} - 1000a + a = 10\overline{abc} - 999a$ (and a similar relationship holds between $\overline{cab}$ and $\overline{bca}$), it follows that the three numbers are each divisible by 27 if and only if $\overline{abc}$ is divisible by 27. Allowing for leading zeroes, there are $\boxed{38}$ three-digit multiples of 27.

45. We can do this problem using the principle of inclusion and exclusion. We first know that the total number of ways to get from A to B without worrying about going through X, Y, or Z is $\binom{12}{6}$. We can then count the number of ways to get from A to X to B which is $\binom{6}{2}\binom{6}{2}$. Similarly for Y we get $\binom{5}{1}\binom{7}{2}$ and for Z we get $\binom{10}{5}\binom{2}{1}$. We then have to add back the case where we go from A to X to Z to B and when we go from A to Y to Z to B. The first case has $\binom{6}{2}\binom{4}{1}\binom{2}{1}$ and in the second case there are $\binom{5}{1}\binom{5}{1}\binom{2}{1}$. Thus, the final equation that yields the correct number of paths is the following,

$$\binom{12}{6} - \binom{6}{2}\binom{6}{2} - \binom{5}{1}\binom{7}{2} - \binom{10}{5}\binom{2}{1} + \binom{6}{2}\binom{4}{1}\binom{2}{1} + \binom{5}{1}\binom{5}{1}\binom{2}{1}$$

This evaluates to $\boxed{260}$.

46. Let point (4,9) be point A and (12,4) be point B. Suppose that the shortest path goes to the y axis first. Then any path P that goes from a point on the y axis to the x axis to point B can be reflected about the y axis so that length of the path is conserved. In fact, we can reflect point B across the y axis to point C, (-12,4), and any path that goes from point A to a point on the y axis to the new point C has the same length as the reflection of any path that goes from point A to the same point on the y axis to the point B. Similarly, after we reach a point on the y axis, we must visit a point on the x axis. Once again, we may reflect point C over the x axis to point (-12,-4) with the same logic. Indeed, the question is identical to finding the shortest length of the path from point (4,9) to point (-12,-4), crossing the x and y axis. Note that the shortest length is just the line segment that connects both points, which clearly passes through both axes. Then the distance by distance formula is just $\sqrt{(4-(-12))^2 + (9-(-4))^2} = \sqrt{16^2 + 13^2} = \sqrt{256 + 169} = \sqrt{425} = x$. Then $x^2 = \boxed{425}$.

47. We know that $2016^2 = 2^{10} \cdot 3^4 \cdot 7^2$, so $k = (10+1) \cdot (4+1) \cdot (2+1) = 165$. Notice for every d_i other than 2016, there is a d_j such that the product of d_i and d_j is 2016^2. Now consider the sum $\dfrac{1}{d_i + 2016} + \dfrac{1}{d_j + 2016}$:

$$\begin{aligned}
\frac{1}{d_i + 2016} + \frac{1}{d_j + 2016} &= \frac{1}{d_i + 2016} + \frac{1}{\frac{2016^2}{d_i} + 2016} \\
&= \frac{1}{d_i + 2016} + \frac{d_i}{2016d_i + 2016^2} \\
&= \frac{2016}{2016d_i + 2016^2} + \frac{d_i}{2016d_i + 2016^2} \\
&= \frac{1}{2016}
\end{aligned}$$

Among the 165 divisors of 2016^2 there are $\dfrac{165-1}{2} = 82$ pairs of such d_i and d_j. Therefore, the desired sum is

$$82 \cdot \frac{1}{2016} + \frac{1}{2016 + 2016} = \frac{165}{4032}$$

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48. An important concept used to solve this problem is *recursion*, where one term of a sequence is determined by the previous terms of that sequence. Let P_r be the probability that a red yarn ball and all of its descendants become blue; we similarly define P_y and P_b. Now we look at the behaviour of each colour ball. A red ball has probability $\frac{5}{12}$ of becoming two red balls, $\frac{1}{3}$ of becoming one red ball and one yellow ball, and $\frac{1}{4}$ of becoming one blue ball. Because balls are independent, we can multiply their probabilities of locking into blue. Moreover, The probability that a certain initial state will lock into blue should be the same as the weighted sum of the probabilities that all of its descendent states will lock into blue. Thus we have $P_r = \frac{5}{12}P_r^2 + \frac{1}{3}P_rP_y + \frac{1}{4}P_b$. Similarly, $P_y = \frac{1}{2}P_y^2 + \frac{1}{4}P_y + \frac{1}{4}P_b$. Blue always stays blue, so $P_b = 1$. Evaluating this system of equations for P_r produces the extraneous solution $P_r = 1$ and the solution $P_r = \boxed{1 - \frac{\sqrt{10}}{5}}$.

49. Let Q be the midpoint of BC. Since $\triangle BMQ \sim \triangle BAC$ with a $1:2$ ratio, $QM = AC/2 = 6$. Similarly, $QN = BD/2 = 9$. Then, from right triangle MQN, we have that $MN = \sqrt{9^2 + 6^2} = \sqrt{117}$. Finally, from right triangle MNP, $MP = \sqrt{117 - 36} = \boxed{9}$.

50. Using the formula $AI^2 = \frac{bc(s-a)}{s}$, where s denotes the semi-perimeter of ABC, we compute $AI^2 = 65$, $BI^2 = 52$, and $CI^2 = 80$. Then, using Heron's Formula, we deduce the area K satisfies $16K^2 = 2(AI^2BI^2 + BI^2CI^2 + CI^2AI^2) - AI^4 - BI^4 - CI^4 = 12151 \implies K^2 = \boxed{\frac{12151}{16}}$.