Joe Holbrook Memorial Math Competition

8th Grade Solutions

October 9th, 2016

1. $$2 + (0 - (1 \cdot 6(2^{0 \cdot \frac{1}{6}}))) = 2 + (0 - (1 \cdot 6(2^0))) = 2 + (0 - (1 \cdot 6)) = 2 - 6 = \boxed{-4}.$$

2. By the formula that says the interior angle of a regular n-sided polygon can be found by $\frac{180 \cdot (n-2)}{n}$, we can see that an 8-sided polygon (octagon) has angles of 135 degrees and a 5-sided polygon (pentagon) has angles of 108 degrees. Therefore, $8 + 5 = \boxed{13}$.

3. Four years pass between Kelvin's 4th and 8th grades, which means that his scores improved 4 times. During this period, his score increased by $43 - 31 = 12$ points. Since he increased by an equal amount every year, we divide the total increase by the total time to get $\frac{12}{4} = 3$. This means that in 7th grade he scored 3 fewer points than in 8th grade, or $43 - 40 = \boxed{40}$.

4. Since doubling any number gives us an even number, we have to work backwards. On Friday, Kelvin had $\frac{48}{2} = 24$ lilypads. On Thursday, he had $\frac{24}{2} = 12$ lilypads. On Wednesday, he had $\frac{12}{2} = 6$ lilypads. On Tuesday, he had $\frac{6}{2} = 3$ lilypads, making our day $\boxed{\text{Tuesday}}$.

5. The value of $f(\pi^2)$ is simply $\pi^2 + 1$, and $g(\pi^2 + 1) = \lfloor \pi^2 + 1 \rfloor = 10$, as π^2 lies between 9 and 10. Thus, $h(\pi^2) = \boxed{10}$.

6. Using Vieta's formula, the sum is $\frac{-(-2)}{1} = \boxed{2}$.

7. 54 flips $= 18 \cdot 5 = 90$ flops. 90 flops $= 10 \cdot 14 = \boxed{140}$ flaps.

8. There is a probability of $\frac{1}{3}$ of pulling out the letter B first; then a probability of $\frac{1}{2}$ of pulling out C; the letter A then has a $\frac{1}{1}$ chance of being selected. Multiplying the fractions together gives a total probability of $\boxed{\frac{1}{6}}$.

9. Arthur ran 40 meters in the first 5 seconds. He only has to run for $\frac{100-40}{3} = \frac{60}{3} = 20$ more seconds. Sunny ran for 32 meters in the first 8 seconds. That means that in $25 - 8 = 17$ seconds, he must run 68 meters, which is an average speed of $\frac{68}{17} = \boxed{4}$ m/s.

10. The number of permutations disregarding the repeated alphabat is $5! = 120$. However, the letter M is repeated twice, thus the number should be divided by $2!$, yielding $\boxed{60}$ as the answer.

11. Note that 2^4 has a units digit of 6. Since $2^{2016} = (2^4)^{504} = 6^{504}$, and every power of 6 ends in 6, we know 2^{2016} has a units digit of 6. Also note that $3^4 = 81$ has a units digit of 1. Since $3^{2016} = (3^4)^{504} = 81^{504}$, we know 3^{2016} has a units digit of 1. Our answer is therefore $1 + 6 = \boxed{7}$.

12. Using the common area formulas for both figures: The square's side s is the solution to $s^2 = 25$, and is therefore 5. For the equilateral triangle, $\frac{s^2 \cdot \sqrt{3}}{4} = 9\sqrt{3}$, so the side length is 6. The difference between the two is $6 - 5 = \boxed{1}$.

13. For $n < 7$, $n! < 1000$, and thus, we must start our search from $n = 7$. Recall that $7! = 5040$ and $7^3 = 343$, so that $7! - 7^3 = 5040 - 343 = 4697 > 2016$. Hence, the least such n is $\boxed{7}$.

14. We note that $\frac{1}{2}+\frac{1}{3}+\frac{1}{9}=\frac{17}{18}$. Therefore, there will be $\frac{1}{18}$ of the coins left after the division. Since there was 1 coin left after Dennis lent the group a coin, there must have been 18 coins in total after the loan. Since we're looking for the original number of coins, we subtract 1 and get $\boxed{17}$.

15. Since $\frac{x}{x+2} < \frac{61}{64}$, multiplying both sides of the inequality by $64(x+2)$ yields $64x < 61(x+2)$, which can be simplified to $3x < 122$, then the largest integer value for x would be $\boxed{40}$.

16. Jake makes 12 cakes per half hour, or 24 per hour. Together, the brothers make 84 cakes in two hours, or 42 per hour. Jake makes 24 of these, leaving Zach to make $42 - 24 = 18$ per hour. In four hours, he makes $18 \cdot 4 = \boxed{72}$.

17. The volume of a cylinder is $\pi r^2 h$, where r is the radius and h is the height. If the new cheesecake is 44% greater in volume, that means that its volume is 144% of the original cheesecake, or the ratio of the two volumes is $\frac{144}{100}=\frac{36}{25}$. If we call the new radius r, then we have that $\frac{\pi r^2 h}{\pi \frac{3}{2}^2 h}=\frac{r^2}{\frac{3}{2}^2}$ also equals this value. Taking the square root of both sides, we have $\frac{r}{\frac{3}{2}}=\frac{6}{5}$, or $r=\boxed{\frac{9}{5}}$.

18. Numbers with an odd number of factors are perfect squares, since the number's square root only counts as a factor once. Numbers with exactly 3 factors must be perfect squares of prime numbers, because in that case the only factors would be 1, itself, and its square root. The square root of 2016 is slightly less than 45 ($45^2 = 2025$), so we must consider the squares of all prime numbers less than 44. These numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, and 43, for a total of $\boxed{14}$ numbers.

19. We can see that the quadrilateral $XABY$ is a trapezoid, and since XY is tangent to circles A and B, $\angle XYB$ and $\angle YXA$ are both right. Since the sum of the angles in a trapezoid is 360 degrees, $\angle XAB + \angle YBA = 180$. We can see that $\triangle XAC$ and $\triangle YBC$ are isosceles, as two of their sides are radii. If we let $\angle XAC = \angle XAB = \alpha$, then $\angle YBC = \angle YBA = 180 - \alpha$. Next, $\angle ACX = \frac{180-\alpha}{2}$, and $\angle BCY = \frac{180-(180-\alpha)}{2} = \frac{\alpha}{2}$. Since $\angle XCY = 180 - (\angle ACX + \angle BCY)$, $\angle XCY = 180 - 90 = \boxed{90}$ degrees.

20. The probability of the test being positive is
$$\frac{1}{150}\cdot\frac{96}{100}+\frac{149}{150}\cdot\frac{4}{100}=\frac{692}{15000}.$$
The probability of a person having the virus and testing positive is
$$\frac{1}{150}\cdot\frac{96}{100}=\frac{96}{15000}.$$
Thus, the probability that Zach has the muggy virus given that he tested positive is
$$\frac{\frac{96}{15000}}{\frac{692}{15000}}=\boxed{\frac{24}{173}}.$$

21. Let's call our three non-negative integers x, y, z. We are given that $x + y = 10 - z$, or $x + y + z = 10$. This can be translated into a stars-and-bars problem, with 10 stars and 2 bars. As a result, our answer is $\binom{12}{2}=\frac{12\cdot 11}{2}=\boxed{66}$.

22. The largest square would have the diameter of the circle as one of its diagonals (it would have length 12), then the side lengths of the square would be $\frac{12}{\sqrt{2}}$, and squaring that yields $\boxed{72}$.

23. From the similarity, $\angle XYZ = 30°$, and $\frac{YZ}{BC}=\frac{XY}{AB} \implies YZ = 2$. Using the sine area formula in XYZ, we see that $[XYZ] = \frac{1}{2}\cdot\sin 30°\cdot 2\cdot 3 = \boxed{\frac{3}{2}}$.

24. Squaring the expression simplifies to 34. As the expression is the sum of two positive quantities, it must itself be positive, so the answer is $\boxed{\sqrt{34}}$.

25. The best way to approach this problem is to consider all of the ways to satisfy the given conditions with pairs of numbers. Notice that there are only two perfect squares from 1-8: 1 and 4. This leaves two remaining triangular numbers, 3 and 6, since 1 cannot be used again.

    This leaves us with the digits 2, 5, 7, and 8. Of these, 2, 5, and 7 are prime. This means that 8 must be one of the first two digits. The other number that must be one of the first two digits must share a factor with 8 other than 1. The only number that satisfies these conditions is 2. Therefore, 5 and 7 are the next two digits, which checks out because they are both prime.

    The four pairs of numbers are (8 2), (5 7), (1 4), and (6 3), respectively. To get the smallest possible 8-digit number, simply list the smaller digit in each pair first. This gives a result of $\boxed{28571436}$.

26. Sort the numbers in groups of two, $(1^2-3^2)+(5^2-7^2)+\cdots+(65^2-67^2)$, then writing each as a difference of squares yields, $(1+3)(1-3)+(5+7)(5-7)+\cdots+(65+67)(65-67)$. This can then be rewritten as $-2(1+3+5+7+...+65+67)$. The sum of the arithmetic series $1+3+\cdots+67$ is 1156, so $-2(1156)=\boxed{-2312}$.

27. Call the number of cheesecakes c. We see that $c-1$ is a multiple of both 4 and 5, and so is a multiple of 20 as well. Now we have that for some positive integers r and s, $c=1+20r$ and $c=2+7s$. Equating the two equations gives us $20r=1+7s$. We now appeal to modular arithmetic, taking the equation (mod 7): $6r\equiv 1\rightarrow -r\equiv 1\rightarrow r\equiv 6$. Thus, r must be 6 more than a multiple of 7. $r=6$ gives us $c=121$, which is too small; the next one must be congruent modulo 4, 5, and 7, so must be $121+4\cdot 5\cdot 7=\boxed{261}$.

28. Since $y=2x+3$, substitute that value into the quadratic. After rearranging, this gives the quadratic $\frac{5}{7}(x^2)-\frac{40}{7}x+5=0$. We are looking for the roots of this equation. This factors as $\frac{5}{7}\cdot(x-7)\cdot(x-1)=0$, so the roots are $x=7$ and $x=1$. Plugging in these values into the line equation yields $y=17$ if $x=7$ and $y=5$ if $x=1$. Using the distance between two points equation gives the distance to be $\boxed{6\sqrt{5}}$.

29. Each road leads out of the two cities it intersects. Thus, counting the total number of roads leading out of the cities gives us twice as many roads that are actually in DrizzleLand. The four largest cities have a total of $4\cdot 7=28$ roads, and the eight smallest cities have $8\cdot 5=40$ roads. Thus, the total number of roads in DrizzleLand is $\frac{28+40}{2}=\boxed{34}$.

30. First we notice that the first three terms sum to 7. For every three term sequence after, taking modulo 7, we can see that every sequence sums to $1+2+4$. Thus, every three term segment leaves no remainder when divided by 7. Since the entire sequence has 2016 terms which is a multiple of 3, a whole number of three-term sequences are contained within the series and the sum leaves a remainder of $\boxed{0}$ when divided by 7.

31. In a standard 8 by 8 chess board, there are 9 horizontal lines and 9 vertical lines, counting the borders. A rectangle is made of 2 vertical lines and 2 horizontal lines. There are $\binom{9}{2}=36$ ways to select 2 horizontal lines, and $\binom{9}{2}=36$ ways to select 2 vertical lines. These are independent events, so multiplying gives us $36^2=\boxed{1296}$.

32. Factoring out the 7 and rationalizing the denominators yields $7(\frac{\sqrt{2}-\sqrt{1}}{2-1}+\frac{\sqrt{3}-\sqrt{2}}{3-2}+\cdots+\frac{\sqrt{49}-\sqrt{48}}{49-48})=7(\sqrt{49}-\sqrt{1})=\boxed{42}$.

33. By the *Chicken McNugget Theorem*, the greatest number that one could not obtain by adding positive multiples of relatively prime numbers m and n is $mn-m-n$. Plugging in $m=5$ and $n=12$, our desired number is $60-12-5=\boxed{43}$. The generalised form is called the *Frobenius Coin Problem*. Google it!

    But what if you've never heard of the theorem? Then we can reason our way to the solution of this problem. Firstly, we know that any number ending in 5 or 0 is attainable (they are all multiples of 5). We also know that any number ending with 2 or 7 starting with 12 can be attainable (they are 12 plus a multiple of 5). Following this line of reasoning, any number ending with 4 or 9 starting with 24; 6 and 1 starting with 36; 8 and 3 starting with 48. We've hit all possible ending digits, so every number after this point is achievable. The last new ones digits attained were 8 and 3, so we look for the greatest one less than 48; that number is $\boxed{43}$.

34. We use casework on the number of dogs that die (0, 2, or 4). The probability that 0 dogs die is $\left(\frac{1}{5}\right)^5$. The probability that 2 dogs die is $\binom{5}{2}\cdot\left(\frac{4}{5}\right)^2\cdot\left(\frac{1}{5}\right)^3$. Finally the probability that 4 dogs die is $\binom{5}{4}\cdot\left(\frac{4}{5}\right)^4\cdot\left(\frac{1}{5}\right)^1$. Summing up yields a probability of $\boxed{\frac{1441}{3125}}$.

35. Let's split our solution into two cases: when the identical digit is 7, and when the identical digit is not 7. If the identical digit is 7, there are 5 choices for the position of the second 7, and the remaining 4 positions can be filled in $9 \times 8 \times 7 \times 6$ ways, for a total of $9 \times 8 \times 7 \times 6 \times 5 = 15120$ numbers. If the identical digit is not 7, there are 9 choices for this digit, $\binom{5}{2} = 10$ ways to place these 2 digits in the remaining 5 positions, leaving $8 \times 7 \times 6$ ways to fill in the 3 remaining positions, for a total of $10 \times 9 \times 8 \times 7 \times 5 = 30240$ numbers. Adding these gives a total of $\boxed{45360}$ numbers.

36. We are not given the dimensions of rectangle $ABCD$, so this should suggest that the exact dimensions are not important, hinting at a general solution that applies for every rectangle. Then, let us consider an arbitrary point O in a rectangle $ABCD$. Construct EF through O such that E lies on AB, F lies on CD, and $EF \parallel AD$. Similarly, construct GH through O such that G lies on AD, H lies on BC, and $GH \parallel AB$. Notice that $AC = BH$, $GD = CH$, $AE = DF$, and $EB = FC$. Then we can write expressions for the distance between O and each of the rectangle's vertices in terms of these lengths.

$$OA^2 = AG^2 + AE^2$$
$$OB^2 = EB^2 + BH^2$$
$$OC^2 = FC^2 + CH^2$$
$$OD^2 = FD^2 + GD^2$$

From this, we realize that $OA^2 + OC^2 = OB^2 + OD^2$ for any generic rectangle and any arbitrary O. Then, we can simply apply this fact back to the original problem, where we are given both OA and OC.

$$OB^2 + OD^2 = OA^2 + OC^2 = 4^2 + 11^2 = \boxed{137}.$$

37. Since a, b, c, d are distinct digits, there are $10 \times 9 \times 8 \times 7 = 5040$ "cool" numbers. Now, the repeating decimal $0.abcd$ can also be written as $\frac{1000a + 100b + 10c + d}{9999}$. Notice that every digit m has a unique digit $n \neq m$ such than $m + n = 9$. Therefore, for all "cool" numbers $P = 0.abcd$, there is another "cool" number $Q = 0.efgh$ such that $P + Q = \frac{1000a + 100b + 10c + d}{9999} + \frac{1000e + 100f + 10g + h}{9999} = \frac{1000(a + e) + 100(b + f) + 10(c + g) + (d + h)}{9999} = \frac{9999}{9999} = 1$. Since there are 5040 "cool" numbers, there are $\frac{5040}{2} = 2520$ such pairs that add up to 1, so our sum is $\boxed{2520}$.

38. There are 4 ways for the distinct prime divisors to add up to 18. The 4 ways are (7,11), (5,13), (2,3,13), and (2,5,11).
- $(7, 11) : 7 * 11 = 77$. (1 number)
- $(5, 13) : 5 * 13 = 65, 5^2 * 13 = 325$. (2 numbers)
- $(2, 3, 13) : 2 * 3 * 13 = 78; 2^2 * 3 * 13 = 156, 2 * 3^2 * 13 = 234, 2^3 * 3 * 13 = 312$. (4 numbers)
- $(2, 5, 11) : 2 * 5 * 11 = 110; 2^2 * 5 * 11 = 220$. (2 numbers) There are 1+2+4+2 = $\boxed{9}$ numbers less than 350 that have distinct prime divisors sum up to 18.

39. Let $S = \frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \frac{7}{16} + \ldots$. Then we know that $\frac{1}{2}S = \frac{1}{4} + \frac{3}{8} + \frac{5}{16} + \ldots$. Subtracting yields that

$$\frac{1}{2}S = \frac{1}{2} + \frac{2}{4} + \frac{2}{8} + \frac{2}{16} + \ldots$$

Hence we have,

$$\frac{1}{2}S = \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right)$$

The expression in the parenthesis is a geometric series with starting term $\frac{1}{2}$ as well as a common ratio of $\frac{1}{2}$ Thus the expression in the parentheses evaluates to $\frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$. Multiplying the overall equation by 2, we have $S = 2 \cdot (\frac{1}{2} + 1) = \boxed{3}$.

40. Because 810000 is divisible by 27, the three numbers are divisible by 27 if and only if $\overline{abc}, \overline{cab}$, and $\overline{bca}$ are all divisible by 27. We consider when one of the three is divisible by 27 by expressing one in terms of another. Since

$\overline{bca} = 10 \cdot \overline{abc} - 1000a + a = 10\overline{abc} - 999a$ (and a similar relationship holds between $\overline{cab}$ and $\overline{bca}$), it follows that the three numbers are each divisible by 27 if and only if $\overline{abc}$ is divisible by 27. Allowing for leading zeroes, there are $\boxed{38}$ three-digit multiples of 27.

41. We can do this problem using the principle of inclusion and exclusion. We first know that the total number of ways to get from A to B without worrying about going through X, Y, or Z is $\binom{12}{6}$. We can then count the number of ways to get from A to X to B which is $\binom{6}{2}\binom{6}{2}$. Similarly for Y we get $\binom{5}{1}\binom{7}{2}$ and for Z we get $\binom{10}{5}\binom{2}{1}$. We then have to add back the case where we go from A to X to Z to B and when we go from A to Y to Z to B. The first case has $\binom{6}{2}\binom{4}{1}\binom{2}{1}$ and in the second case there are $\binom{5}{1}\binom{5}{1}\binom{2}{1}$. Thus, the final equation that yields the correct number of paths is the following,

$$\binom{12}{6} - \binom{6}{2}\binom{6}{2} - \binom{5}{1}\binom{7}{2} - \binom{10}{5}\binom{2}{1} + \binom{6}{2}\binom{4}{1}\binom{2}{1} + \binom{5}{1}\binom{5}{1}\binom{2}{1}$$

This evaluates to $\boxed{260}$.

42. A hexagon can be divided into 6 equilateral triangles, each of side length 4. Using the equilateral triangle area formula, we get that the area of each equilateral triangle has an area of $4\sqrt{3}$. Additionally, as the small triangles between consecutive hexagons have 30 degree angles, we can find that the side length of the next hexagon is $2\sqrt{3}$, and using similar methods, we find that the 6 equilateral triangles that make up this hexagon each have an area of $3\sqrt{3}$. In order for a region to be in an odd number of hexagons, it is the alternating area between hexagons. Therefore, the answer is the area of hexagon $A_1B_1C_1D_1E_1F_1$ minus $A_2B_2C_2D_2E_2F_2$ plus $A_3B_3C_3D_3E_3F_3$ minus... Thus, you will get: $24\sqrt{3} - 18\sqrt{3} + ...$, and this infinite geometric series will get the answer of $\boxed{\frac{96\sqrt{3}}{7}}$.

43. Let point (4,9) be point A and (12,4) be point B. Suppose that the shortest path goes to the y axis first. Then any path P that goes from a point on the y axis to the x axis to point B can be reflected about the y axis so that length of the path is conserved. In fact, we can reflect point B across the y axis to point C, (-12,4), and any path that goes from point A to a point on the y axis to the new point C has the same length as the reflection of any path that goes from point A to the same point on the y axis to the point B. Similarly, after we reach a point on the y axis, we must visit a point on the x axis. Once again, we may reflect point C over the x axis to point (-12,-4) with the same logic. Indeed, the question is identical to finding the shortest length of the path from point (4,9) to point (-12,-4), crossing the x and y axis. Note that the shortest length is just the line segment that connects both points, which clearly passes through both axes. Then the distance by distance formula is just $\sqrt{(4-(-12))^2 + (9-(-4))^2} = \sqrt{16^2 + 13^2} = \sqrt{256 + 169} = \sqrt{425} = x$. Then $x^2 = \boxed{425}$.

44. Since x is a positive real number, we can rewrite $x = n + y$, where n is a non-negative integer and $y < 1$ is a non-negative real number. Then, $\lfloor 2x \rfloor + \lfloor 3x \rfloor + \lfloor 4x \rfloor = \lfloor 2(n+y) \rfloor + \lfloor 3(n+y) \rfloor + \lfloor 4(n+y) \rfloor = 9n + \lfloor 2y \rfloor + \lfloor 3y \rfloor + \lfloor 4y \rfloor$. Since the number of expressible integers does not depend on n, let's take a look at the values of $K = \lfloor 2y \rfloor + \lfloor 3y \rfloor + \lfloor 4y \rfloor$. Notice that $0 \leqslant y < 1$ and $0 \leqslant K \leqslant 6$. Now, let's split $[0, 1)$ into 6 regions: $R_1 = [0, \frac{1}{4}), R_2 = [\frac{1}{4}, \frac{1}{3}), R_3 = [\frac{1}{3}, \frac{1}{2}), R_4 = [\frac{1}{2}, \frac{2}{3}), R_5 = [\frac{2}{3}, \frac{3}{4}), R_6 = [\frac{3}{4}, 1)$. Now consider y being in each of these six regions: $y \epsilon R_1 \to K = 0$; $y \epsilon R_2 \to K = 1$; $y \epsilon R_3 \to K = 2$; $y \epsilon R_4 \to K = 4$; $y \epsilon R_5 \to K = 5$; $y \epsilon R_6 \to K = 6$. Therefore, out of 9 total possible values of K (as integers from 0 to 8), 6 are possible. Therefore for $n = 0$ to $n = 10$, there are $11 \times 6 = 66$ possible values. However, when $n = 0$ and $K = 0$, our value is 0, which is not a positive integer. So there are $66 - 1 = 65$ possible values. But for $n = 11$, $K = 0$ and $K = 1$ work, adding 2 more solutions for a total of $\boxed{67}$ possible values.

45. Since $2016^2 = 2^{10} \cdot 3^4 \cdot 7^2$, so $k = (10+1) \cdot (4+1) \cdot (2+1) = 165$. Notice for every d_i other than 2016, there is a d_j

such that the product of d_i and d_j is 2016^2. Now consider the sum $\dfrac{1}{d_i+2016}+\dfrac{1}{d_j+2016}$:

$$\begin{aligned}\frac{1}{d_i+2016}+\frac{1}{d_j+2016} &= \frac{1}{d_i+2016}+\frac{1}{\frac{2016^2}{d_i}+2016}\\ &= \frac{1}{d_i+2016}+\frac{d_i}{2016d_i+2016^2}\\ &= \frac{2016}{2016d_i+2016^2}+\frac{d_i}{2016d_i+2016^2}\\ &= \frac{1}{2016}\end{aligned}$$

Among the 165 divisors of 2016^2 there are $\dfrac{165-1}{2}=82$ pairs of such d_i and d_j. Therefore, the desired sum is $82\cdot\dfrac{1}{2016}+\dfrac{1}{2016+2016}=\boxed{\dfrac{165}{4032}}$.

46. The rightmost point of this ellipse is the point $(3,0)$. When rotated, this point will be on the line $y=x$ at a distance of 3 from the origin; this point is $(\frac{3}{\sqrt{2}},\frac{3}{\sqrt{2}})$. Plugging this into the general equation, we have $(a+b+c)(xy)=(a+b+c)(\frac{9}{2})=32$, or $a+b+c=\boxed{\dfrac{64}{9}}$.

47. Let's rewrite our equality:
   $x^2+17-16y=-y^2+12x+13$; subtracting and completing the square for both x and y yields
   $x^2-12x+36-36+y^2-16y+64-64+17-13=0$
   $(x-6)^2+(y-8)^2=96$
   $[(x-6)+(y-8)]^2=96+2\times(x-6)(y-8)$
   $(x-6)+(y-8)=\sqrt{96+2\times(x-6)(y-8)}$
   $\dfrac{(x-6)+(y-8)}{2}=\dfrac{\sqrt{96+2\times(x-6)(y-8)}}{2}$
   By the AM-GM inequality, $\dfrac{(x-6)+(y-8)}{2}\geqslant\sqrt{(x-6)(y-8)}$, so we have:
   $\dfrac{\sqrt{96+2\times(x-6)(y-8)}}{2}\geqslant\sqrt{(x-6)(y-8)}$.
   $96+2\times(x-6)(y-8)\geqslant 4\times(x-6)(y-8)$
   $48\geqslant(x-6)(y-8)$
   Therefore, $(x-6)+(y-8)=\sqrt{96+2\times(x-6)(y-8)}\leqslant\sqrt{96+2\times 48}=\sqrt{192}=8\sqrt{3}$, so $x+y-14\leqslant 8\sqrt{3}$, so $x+y\leqslant 14+8\sqrt{3}$. Therefore, the maximum value of $x+y$ is $\boxed{14+8\sqrt{3}}$, which occurs when $x=6+4\sqrt{3}$ and $y=8+4\sqrt{3}$.

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48. et Q be the midpoint of BC. Since $\triangle BMQ\sim\triangle BAC$ with a $1:2$ ratio, $QM=AC/2=6$. Similarly, $QN=BD/2=9$. Then, from right triangle MQN, we have that $MN=\sqrt{9^2+6^2}=\sqrt{117}$. Finally, from right triangle MNP, $MP=\sqrt{117-36}=\boxed{9}$.

49. Using the formula $AI^2=\dfrac{bc(s-a)}{s}$, where s denotes the semi-perimeter of ABC, we compute $AI^2=65$, $BI^2=52$, and $CI^2=80$. Then, using Heron's Formula, we deduce the area K satisfies

$$16K^2=2(AI^2BI^2+BI^2CI^2+CI^2AI^2)-AI^4-BI^4-CI^4=12151 \implies K^2=\boxed{\frac{12151}{16}}.$$

50. Consider $[kn]$, for some positive integer k. Notice by grouping the kn digits into groups of k that $[kn]$ is divisible by $[k]$ (for example, $[6]=111111$ can be written as $111\cdot 1001=[3]\cdot 1001$). Next, confirm the following transformation: $\{m\}\cdot\dfrac{11}{10}=[2m]$.

Thus we want to find the largest n such that $[n]$ divides $[2016] \cdot \frac{10}{11}$. $[n]$ has no factors of 2 or 5, so we can disregard the 10, and we have $11[n]$ divides $[2016]$, and as a result n divides 2016.

Keep in mind that we still need to meet the condition that $[n]$ divides $\{1008\} = 101010\ldots101010$. There are 504 1's, which means that by the divisibility rule for 11, 11 does not divide $\{1008\}$. This means that $[n]$ cannot have 11 as a factor; however, $11 = [2]$, so $[n]$ cannot have an even number of digits, and therefore n is odd. The largest odd factor of 2016 is $\boxed{63}$."