Joe Holbrook Memorial Math Competition

7th Grade Solutions

October 15, 2017

1. It is 2 hours and 36 minutes from 3:00, and as there are 60 minutes in an hour, $2 \cdot 0 + 36 = \boxed{156}$

2. $300 - 60 - 30 - 111 = \boxed{99}$.

3. The battery will recharge at the rate of $2 - 0.5 = 1.5\%$ per minute. Because it needs to recharge 60%, the answer is $60/1.5 = \boxed{40}$.

4. Let x be the number of shots he must make to reach his goal. We have the inequality $\frac{7+x}{15+x} \geq 0.7$, or $x \geq \frac{35}{3}$. Since x must be a positive integer, the minimum valid value for x is $\boxed{12}$.

5. Call the original number of boats x. The condition tells us that $6(x+1) = 9(x-1)$, which can be solved and $x = 5$. Thus the number of people is $6 \cdot (5+1) = \boxed{36}$.

6. There are 31 days in July, so we consider the positive integers less than or equal to 31 that are perfect squares or powers of 2. The perfect squares are $1, 4, 9, 16, 25$, and the powers of 2 are $1, 2, 4, 8, 16$. Thus, Haneul and Julia video-call each other for half an hour on the $1st, 4th, 9th, 16th$, and $25th$, and call for an hour on the $2nd$ and $8th$, so they spend a total of $5 \cdot \frac{1}{2} + 2 \cdot 1 = \boxed{\frac{9}{2}}$ hours communicating.

7. From the Triangle Inequality, we have that $3 < x < 11$. In order to have an obtuse triangle, the sum of the squares of two sides must be less than the square of the longest side. Thus, either $4^2 + x^2 < 7^2$ or $4^2 + 7^2 < x^2$ must be satisfied. The first gives us $x^2 < 33$. The second gives us $x^2 > 65$. Thus, the values of x that satisfy either of these conditions and the Triangle inequality are 4, 5, 9, and 10. This gives us a total of $\boxed{4}$ values.

8. The player makes a total of $15 \cdot 2 + 20 \cdot 3 + 25 \cdot 1 = 115$ points in 60 baskets, so the average number of points per shot is $\frac{15 \cdot 2 + 20 \cdot 3 + 25 \cdot 1}{60} = \boxed{\frac{23}{12}}$.

9. To find how many "good" sandwiches there are, we need to subtract the number of "bad" sandwiches from the total number of possible sandwiches. There are 4 sandwiches with salami and american cheese, and 3 sandwiches with roast beef and baloney, resulting in 7 "bad" sandwiches. The total number of sandwiches is 30 ($\binom{5}{2} \cdot \binom{3}{1} = 10 \cdot 3$). Subtracting $30 - 7 = \boxed{23}$ good sandwiches.

10. There are $7!/2!$ possible ways of rearranging *Tiffany* as there are 7 letters and the *f* is repeated twice. There are $6!$ combinations of rearranging *Tifany* as there are 6 letters and no repeated letters. $7!/2! = 2520$ and $6! = 720$. $2520 - 720 = \boxed{1800}$.

11. The factors of 42 are: $1, 2, 3, 6, 7, 14, 21$, and 42. They pair up and multiply to 42. There are 4 pairs. Therefore, their product is 42^4.

12. By the identity $\text{lcm}(a,b) \cdot \gcd(a,b) = ab$, it follows that $ab = 20 \cdot 10 = \boxed{200}$.

13. Note that $0\Delta a = a\Delta 0 = 0$ if $a \neq 0$. Thus $((2\Delta 6)\Delta 3)\Delta(0\Delta(3\Delta 7)) = \boxed{0}$.

14. Since the sequence starts off with 2 odd terms, every third term will be even. Thus, all the terms that are multiples of three are the ones we are looking for. Among the first 100, thee are $\boxed{33}$ multiples of 3.

15. Between any two circle, there can at most exist 2 intersections. Therefore, we have $2 \cdot \binom{17}{2}$. Between any two line, there can at most exist 1 intersection. Therefore, we have another $\binom{17}{2}$ intersections. Between

a line and a circle, there can exist 1 intersection, for an additional we have $2 \cdot 17^2$intersections. This yields a total of $\boxed{986}$ intersections.

16. In those 5 numbers, there must be at least 2 5's in order to make it the unique mode. In order to minimize the average of the smallest 4 numbers, you want 1 and 2 to be part of the set (you can't have 2 1's since that would mean that 5 is not the unique mode). Now, given these 4 numbers, you need $\boxed{12}$ in order to make the average 5.

17. There are $\binom{6}{3}$ ways to choose 3 socks of the same color, and there are 3 different colors. Meanwhile, there are $\binom{18}{3}$ ways to choose 3 socks. Therefore, the probability is $\frac{3 \cdot \binom{6}{3}}{\binom{18}{3}} = \boxed{5/68}$.

18. Triangles $\triangle ABD$ and $\triangle BDC$ are equilateral since $\frac{120^\circ}{2} = 60^\circ$ so the total area is $2 \cdot \frac{6^2\sqrt{3}}{4} = \boxed{18\sqrt{3}}$.

19. Looking at the hundreds place, the two digits have to be 9 and 1. Now, if the tenth place of the subtrahend is 1, the smallest it can be is 110, and the minuend would have to be more than 894+110=1004. Thus the tens place of the subtrahend is $\boxed{0}$, and so is the product of the digits.

20. Construct the altitude of $\triangle MAN$ onto side MN, call the foot O. We have $OM = BM$ while $ON = ND$. These two equality conditions lead to two sets of congruent triangles, thus $\angle MAN$, which has half of both sets, has a degree measure of $\boxed{45}$ degrees.

21. Since Haneul and Ivy always sit together, we can consider them as one person. Once we have done this, the only "person" Julia can sit next to is the pair Haneul and Ivy. Therefore Julia must be on the end, and she must be next to Haneul and Ivy. This pair can be oriented two ways, and Julia can be on the left or the right, and Ben and David can be oriented two ways, so there are $\boxed{8}$ total permutations.

22. As the given quantity is a difference of squares, it may be factored as $3^{10} - 1 = (3^5 - 1)(3^5 + 1) = 242 \cdot 244$. 242 factors as $2 \cdot 11^2$ and $244 = 2^2 \cdot 61$. Thus, the largest prime divisor of $3^{10} - 1$ is $\boxed{61}$.

23. By properties of $30 - 60 - 90$ degree triangles, the height of the tree must be $6\sqrt{3}$ feet tall. As the triangle formed by the position of the bug, David, and the base of the tree forms another $30 - 60 - 90$ degree triangle, the height of the bug above the ground is $\frac{6}{\sqrt{3}} = \boxed{2\sqrt{3}}$ feet.

24. Listing out possibilities, we see that the 7 cases HHHTT, HHHHT, HHHTH, HHHHH, THHHT, THHHH, TTHHH, are the only possibilities, so the answer is $\boxed{\frac{7}{32}}$.

25. Chord BP is a diameter of Ω as point O lies on BP. As the radius of Ω is 2, the length of BP is 4, and the angle $\angle BCP = 90^\circ$, applying the Pythagorean Theorem on $\triangle BCP$ yields $BC^2 + PC^2 = BP^2$. Plugging in the given values yields $BC^2 = 4^2 - 3^2 = 7 \to BC = \boxed{\sqrt{7}}$.

26. $1200 = 2^4 \cdot 3 \cdot 5^2$, so there are $5 \cdot 2 \cdot 3 = 30$ total factors and $3 \cdot 1 \cdot 2 = 6$ perfect square factors. Therefore, there are $\boxed{24}$ non-perfect square factors of 1200.

27. As $AB = 8$ and $BC = AD = 6$, by the Pythagorean Theorem, it follows that $BD = 10$. As M is the midpoint of BD, DM must be 5. As AN is the height to the base BD in triangle $\triangle ABD$, $\frac{1}{2} \cdot AN \cdot BD = \frac{1}{2} \cdot AB \cdot AD \to AN = \frac{24}{5}$. By the Pythagorean Theorem in triangle $\triangle AND$, $DN^2 = AD^2 - AN^2 \to DN = \frac{18}{5} \to MN = \boxed{\frac{7}{5}}$.

28. Set side BC to be of length x. Then side CD is of length $37.5 - x$, since the sum of the two sides is half the perimeter. Then, $x \cdot 14 = (37.5 - x) \cdot 16$, solving gives us $x = 20$. Thus the area is $20 \cdot 14 = \boxed{280}$

29. If the area of the regular hexagon is twice that of the area of the equilateral triangle, then the area of each equilateral triangle that is formed by connected opposite vertices of the hexagon is a third of the area of the equilateral triangle Neb was supposed to draw. This means that the side length of the equilateral triangle must be $\frac{1}{\sqrt{3}}$ of the original equilateral triangle, since the ratio of side lengths is the square root of the ratio of areas. Therefore, the side length is $\boxed{2\sqrt{3}}$.

30. Note that $84 = 2^2 \cdot 3 \cdot 7$, $120 = 2^3 \cdot 3 \cdot 5$, and $126 = 2 \cdot 3^2 \cdot 7$. Then, taking the greatest common divisor of each pair of the three values yields $2^2 \cdot 3, 2 \cdot 3 \cdot 7$, and $2 \cdot 3$. Since if an integer divides a corresponding pair of integers, then it must divide the greatest common divisor of the integers, we then sum the total number of divisors of each of the gcds: $3 \cdot 2 = 6, 2 \cdot 2 \cdot 2 = 8$, and $2 \cdot 2 = 4$, so $6 + 8 + 4 = 18$ total factors. However, we are overcounting the numbers that divide all three values. Since the gcd of all three is $2 \cdot 3$, we overcounted $2 \cdot 2 = 4$ divisors twice, for a total of $2 \cdot 4 = 8$ overcounted divisors. Thus, our answer is $18 - 8 = \boxed{10}$.

31. Clearly there are 100 total outcomes. Let m be a positive integer. We will count the number of points Bessie can choose in order to have a line with slope m, and sum over all positive integers m.

    By definition of slope, the points Bessie chooses must be of the form (k, km) for some positive integer k. Since we must have $km \leq 10$, we get $1 \leq k \leq \left\lfloor \frac{10}{m} \right\rfloor$, or $\left\lfloor \frac{10}{m} \right\rfloor$ valid slopes. Noting that Bessie cannot choose any point such that the line has a slope greater than 10, we sum over the first 10 positive integers to get $10 + 5 + 3 + 2 + 2 + 1 + 1 + 1 + 1 + 1 = 27$ valid lattice points. Dividing over the total number of cases gives the desired answer of $\boxed{\frac{27}{100}}$.

32. There are $\binom{20}{2} = 190$ highways total. Consider how many highways that are needed in order to connect every town. Without loss of generality, let town A and B be connected. In order for town C to be connected to both of them, town C only needs to be connected to A or B, but not both. Repeat for all the remaining towns, and you will get 19 required highways. Therefore, $\boxed{171}$ of them can be closed.

33. Each units digit of 2017^x is 7, 9, 3, or 1 depending what the remainder of x is when divided by 4. Since $x = 2016^{2015}$ in this case and 2016^{2015} divided by 4 is 0 as 2016 is a multiple of 4, the units digit is $\boxed{1}$.

34. By power of a point, we know $PB \cdot PC = PA^2$.Since AC is a diameter, and PA is a tangent, we know that $AC \perp PA$. Thus we can use the Pythagorean theorem to solve for PC.
    We have $PA^2 + AC^2 = PC^2$. Substituting $PB \cdot PC = 16 \cdot PC$ for PA^2, and rearranging, we get $PC^2 - 16PC - 15^2 = 0$. Solving for PC and taking the positive root we get $\boxed{25}$.

35. Since $\binom{n}{k} = 0$ if $k < n$, we only consider the terms for which $n \leq k$. Thus, the desired sum is

$$\binom{1}{1} + \left(\binom{2}{1} + \binom{2}{2}\right) + \left(\binom{3}{1} + \binom{3}{2} + \binom{3}{3}\right) + \cdots + \left(\binom{6}{1} + \binom{6}{2} + \cdots + \binom{6}{6}\right).$$

    From the Binomial Theorem,

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n},$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n},$$

$$\binom{n}{1} + \cdots + \binom{n}{n} = 2^n - \binom{n}{0} = 2^n - 1.$$

    Hence, the desired sum is equivalent to

$$\begin{aligned} &(2^1 - 1) + (2^2 - 1) + \cdots + (2^6 - 1) \\ &= 2^1 + 2^2 + \cdots + 2^6 - 6 \\ &= \frac{2^7 - 1}{2 - 1} - 6 \\ &= 2^7 - 7 = 128 - 7 = 121. \end{aligned}$$

    There are $6 \cdot 6 = 36$ possible distinct rolls, so the expected value of $\binom{x}{y}$ is $\boxed{\frac{121}{36}}$.

36. We aim to find many perpendicular lines. EG is normal to $BDHF$. Call the intersection point I. Now we find the distance from point I to BH. That distance is clearly half of the height from point F to side BH in the triangle BHF, which is $\frac{\sqrt{6}}{3}$. The answer is thus $\boxed{\frac{\sqrt{6}}{6}}$.

37. As $x^2+x+1=0$, $x^3=1$. Thus, $x^{200}=(x^{66})^3\cdot x^2=x^2$ and $x^{100}=(x^{33})^3\cdot x=x$, so $x^{200}+x^{100}+1=x^2+x+1=\boxed{0}$.

38. It is evident that the circle with radius X is surrounded by rest of the circles. Connecting the pairwise centers of the circles with known radius, the side lengths of the triangle formed are $16,17,17$. Dropping the altitude to the base of this isosceles triangle, the height of the triangle to the base is $\sqrt{17^2-8^2}=15$ by Pythagoras. Connect the vertices of the triangle to the center of the circle with radius X. By using pythagorean theorem, we know that the $\sqrt{(8+X)^2-8^2}+X+9=15$. Simplifying this formula gives $\sqrt{X^2+16X}=6-X$. Squaring both sides yields $X^2+16X=X^2-12X+36\implies 28X=36\implies X=\boxed{\frac{9}{7}}$.

39. Let the center of the circle be O. WLOG, we consider $\triangle OPQ$. Let the foot of the perpendicular from O to BC be M, so $BM=\frac{1}{2}BQ=3$. Since $OP=5$, from right triangle OBM, we have $OM=4$. Identically, if the feet of the perpendiculars from O to BC and CA are L and N, then $OL=ON=4$. Thus, O is the incenter of $\triangle ABC$.

Let $AB=x$. Recall that the area of $\triangle ABC$ is rs, where r is the inradius and s is the semiperimeter, and is also $\frac{24x}{2}=12x$. Thus,

$$\begin{aligned}
[\triangle ABC]=rs=4\frac{\left(24+x+\sqrt{24^2+x^2}\right)}{2}&=2\left(24+x+\sqrt{576+x^2}\right),\\
[\triangle ABC]&=12x,\\
2\left(24+x+\sqrt{576+x^2}\right)&=12x,\\
24+x+\sqrt{576+x^2}&=6x,\\
\sqrt{576+x^2}&=5x-24\\
576+x^2&=25x^2-240x+576,\\
240x&=24x^2,\\
240&=24x,\\
x&=10.
\end{aligned}$$

Hence, $AB=\boxed{10}$.

40. Let E be the expected value of P. The expected value of any roll of the die is $\frac{1+2+3+4+5+6}{6}=\frac{7}{2}$, so $a=b=c=d=\frac{7}{2}$. It follows that $E(1)=7, E(2)=\frac{49}{4}, E(3)=7, E(4)=\frac{49}{4}$.

Since P has degree at most 3, E does as well, so its third finite differences are constant. Note that its first finite differences are

$$E(2)-E(1), E(3)-E(2), E(4)-E(3), E(5)-E(4), ...,$$

its second finite differences are

$$E(3)-2E(2)+E(1), E(4)-2E(3)+E(2), E(5)-2E(4)+E(3), ...,$$

so its third finite differences are

$$E(4)-3E(3)+3E(2)-E(1), E(5)-3E(4)+3E(3)-E(2), ...$$

Thus,

$$\begin{aligned}
E(4)-3E(3)+3E(2)-E(1)&=E(5)-3E(4)+3E(3)-E(2),\\
E(5)&=4E(4)-6E(3)+4E(2)-E(1)\\
E(5)&=4\left(\frac{49}{4}\right)-6(7)+4\left(\frac{49}{4}\right)-7\\
&=49-42+49-7=\boxed{49}.
\end{aligned}$$