Joe Holbrook Memorial Math Competition

8th Grade Solutions

October 15, 2017

1. Since addition and subtraction come before division and mutiplication, $2 + 5 \cdot 2 - 5 = (2 + 5) \cdot (2 - 5)$. Now, since operations are performed right to left, $(2 + 5) \cdot (2 - 5) = 7 \cdot 3 = \boxed{21}$.
2. There are 1000 meters in a kilometer, so it takes Malik $12.21 \cdot \frac{5000}{100} = 12.21 \cdot 50 = \boxed{611.5}$ seconds to finish the race.
3. Let x be the number of shots he must make to reach his goal. We have the inequality $\frac{7+x}{15+x} \geq 0.7$, or $x \geq \frac{35}{3}$. Since x must be a positive integer, the minimum valid value for x is $\boxed{12}$.
4. Let x, y be the number of heads and tails, respectively. The condition fails if and only if $x < 500$ and $y < 500$. Summing the inequalities yields $x + y < 1000$, which is a contradiction. Thus, the probability that this occurs is 100%, so $n = \boxed{100}$.
5. The factors of 42 are: 1, 2, 3, 6, 7, 14, 21, and 42. They pair up and multiply to 42. There are 4 pairs. Therefore, their product is 42^4, so $n = \boxed{4}$.
6. By the identity lcm $(a, b)\cdot$ gcd $(a, b) = ab$, it follows that $ab = 20 \cdot 10 = \boxed{200}$.
7. Since the sequence starts off with 2 odd terms, every third term will be even. Thus, all the terms that are multiples of three are the ones we are looking for. Among the first 100, thee are $\boxed{33}$ multiples of 3.
8. Between any two circle, there can at most exist 2 intersections. Therefore, we have $2 \cdot \binom{17}{2}$. Between any two line, there can at most exist 1 intersection. Therefore, we have another $\binom{17}{2}$ intersections. Between a line and a circle, there can exist 1 intersection, for an additional we have $2 \cdot 17^2$intersections. This yields a total of $\boxed{986}$ intersections.
9. In those 5 numbers, there must be at least 2 5's in order to make it the unique mode. In order to minimize the average of the smallest 4 numbers, you want 1 and 2 to be part of the set (you can't have 2 1's since that would mean that 5 is not the unique mode). Now, given these 4 numbers, you need $\boxed{12}$ in order to make the average 5.
10. There are $\binom{6}{3}$ ways to choose 3 socks of the same color, and there are 3 different colors. Meanwhile, there are $\binom{18}{3}$ ways to choose 3 socks. Therefore, the probability is $\frac{3 \cdot \binom{6}{3}}{\binom{18}{3}} = \boxed{5/68}$.
11. Looking at the hundredth place, the two digits have to be 9 and 1. Now, if the tenth place of the subtrahend is 1, the smallest it can be is 110, and the minuend would have to be more than $894 + 110 = 1004$. Thus the tenth place of the subtrahend is $\boxed{0}$, and so is the product of the digits.
12. Let the foot of the altitude from A onto side MN, be O. We get $OM = BM$ and ON=ND. These two equality conditions lead to two sets of congruent triangles, thus $\angle MAN$, which has half of both sets, has a degree measure of $\boxed{45}$ degrees.
13. The center of the square can be located inside the square of 1 inch inside each tile. Therefore, the probability is $\frac{1^2}{2^2} = \boxed{1/4}$.
14. Chord BP is a diameter of Ω as point O lies on BP. As the radius of Ω is 2, the length of BP is 4, and the angle $\angle BCP = 90^\circ$, applying the Pythagorean Theorem on $\triangle BCP$ yields $BC^2 + PC^2 = BP^2$. Plugging in the given values yields $BC^2 = 4^2 - 3^2 = 7 \rightarrow BC = \boxed{\sqrt{7}}$.

15. As $AB = 8$ and $BC = AD = 6$, by the Pythagorean Theorem, it follows that $BD = 10$. As M is the midpoint of BD, DM must be 5. As AN is the height to the base BD in triangle $\triangle ABD$, $\frac{1}{2} \cdot AN \cdot BD = \frac{1}{2} \cdot AB \cdot AD \to AN = \frac{24}{5}$. By the Pythagorean Theorem in triangle $\triangle AND$, $DN^2 = AD^2 - AN^2 \to DN = \frac{18}{5} \to MN = \boxed{\frac{7}{5}}$.

16. Set side BC to be of length x. Then side CD is of length $37.5 - x$, since the sum of the two sides is half the perimeter. Then, $x \cdot 14 = (37.5 - x) \cdot 16$, solving gives us $x = 20$. Thus the area is $20 \cdot 14 = \boxed{280}$.

17. Note that $84 = 2^2 \cdot 3 \cdot 7$, $120 = 2^3 \cdot 3 \cdot 5$, and $126 = 2 \cdot 3^2 \cdot 7$. Then, taking the greatest common divisor of each pair of the three values yields $2^2 \cdot 3, 2 \cdot 3 \cdot 7$, and $2 \cdot 3$. Since if an integer divides a corresponding pair of integers, then it must divide the greatest common divisor of the integers, we then sum the total number of divisors of each of the gcds: $3 \cdot 2 = 6, 2 \cdot 2 \cdot 2 = 8$, and $2 \cdot 2 = 4$, so $6 + 8 + 4 = 18$ total factors. However, we are overcounting the numbers that divide all three values. Since the gcd of all three is $2 \cdot 3$, we overcounted $2 \cdot 2 = 4$ divisors twice, for a total of $2 \cdot 4 = 8$ overcounted divisors. Thus, our answer is $18 - 8 = \boxed{10}$.

18. Clearly there are 100 total outcomes. Let m be a positive integer. We will count the number of points Bessie can choose in order to have a line with slope m, and sum over all positive integers m.

    By definition of slope, the points Bessie chooses must be of the form (k, km) for some positive integer k. Since we must have $km \leq 10$, we get $1 \leq k \leq \left\lfloor \frac{10}{m} \right\rfloor$, or $\left\lfloor \frac{10}{m} \right\rfloor$ valid slopes. Noting that Bessie cannot choose any point such that the line has a slope greater than 10, we sum over the first 10 positive integers to get $10 + 5 + 3 + 2 + 2 + 1 + 1 + 1 + 1 + 1 = 27$ valid lattice points. Dividing over the total number of cases gives the desired answer of $\boxed{\frac{27}{100}}$.

19. The condition is $2a^2 + b = b^2 + ab + b$, which simplifies to $(a - b)(2a + b) = 0$. Since a and b are different, $2a + b = 0$. $f(2)$ is $4 + 2a + b$, so the desired value is $\boxed{4}$.

20. There are $\binom{20}{2} = 190$ highways total. Consider how many highways that are needed in order to connect every town. Without loss of generality, let town A and B be connected. In order for town C to be connected to both of them, town C only needs to be connected to A or B, but not both. Repeat for all the remaining towns, and you will get 19 required highways. Therefore, $\boxed{171}$ of them can be closed.

21. The diagonal of the kite that bisects the kite must be a diameter, so it has length 10. This forms a right triangle with hypotenuse 10 and a leg with length 6, so the other leg must have length $\boxed{8}$.

22. We claim that $\frac{|x - y| + |y - z| + |z - x|}{2}$ is the numeric value for the largest positive value of the difference between 2 terms of x, y, z.

    To see this, consider WLOG that $x \leq y \leq z$. Then $|x - y| + |y - z| + |z - x| = y - x + z - y + z - x = 2z - 2x = 2(z - x)$, and the result follows.

    Hence, the maximum possible value is $20/2 = \boxed{10}$. This can be attained by letting $x = 10, y = 0, z = 0$.

23. There are 220 ways to choose 3 edges from a cube, which has 12 edges. The 12 edges can be grouped based on their orientations, with each group containing 4 parallel edges. Since the 3 edges have to be pairwise skew, it is imperative to choose one edge from each group. Choose one edge at random from any of the groups. There are 4 possibilities. Two edges from each of the other two groups intersect the selected edge, so choose any of the other two groups and an edge at random. There are 2 possibilities. There is only one edge left from the third group. Thus there are 8 possibilities in total to make the desired 3-edge selection, and the probability is $\boxed{\frac{2}{55}}$.

24. "By power of a point, we know $PB \cdot PC = PA^2$.Since AC is a diameter, and PA is a tangent, we know that $AC \perp PA$. Thus we can use the Pythagorean theorem to solve for PC.
    We have $PA^2 + AC^2 = PC^2$. Substituting $PB \cdot PC = 16 \cdot PC$ for PA^2, and rearranging, we get $PC^2 - 16PC - 15^2 = 0$. Solving for PC and taking the positive root we get $\boxed{25}$.

25. Since $\binom{n}{k} = 0$ if $k < n$, we only consider the terms for which $n \leq k$. Thus, the desired sum is

$$\binom{1}{1} + \left(\binom{2}{1} + \binom{2}{2}\right) + \left(\binom{3}{1} + \binom{3}{2} + \binom{3}{3}\right) + \cdots + \left(\binom{6}{1} + \binom{6}{2} + \cdots + \binom{6}{6}\right).$$

From the Binomial Theorem,

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n},$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n},$$

$$\binom{n}{1} + \cdots + \binom{n}{n} = 2^n - \binom{n}{0} = 2^n - 1.$$

Hence, the desired sum is equivalent to

$$\begin{aligned} &(2^1 - 1) + (2^2 - 1) + \cdots + (2^6 - 1) \\ &= 2^1 + 2^2 + \cdots + 2^6 - 6 \\ &= \frac{2^7 - 1}{2 - 1} - 6 \\ &= 2^7 - 7 = 128 - 7 = 121. \end{aligned}$$

There are $6 \cdot 6 = 36$ possible distinct rolls, so the expected value of $\binom{x}{y}$ is $\boxed{\frac{121}{36}}$.

26. We aim to find many perpendicular lines. EG is normal to $BDHF$. Call the intersection point I. Now we find the distance from point I to BH. That distance is clearly half of the height from point F to side BH in the triangle BHF, which is $\frac{\sqrt{6}}{3}$. The answer is thus $\boxed{\frac{\sqrt{6}}{6}}$.

27.
$$\begin{aligned} &P(\text{gets as many heads as tails}) \\ &= \frac{\text{number of ways to get strictly more heads than tails} + \text{number of ways to get the same number of heads and tails}}{\text{total number of possible coin flip sequences}} \\ &= \frac{\frac{2^{12} - \binom{12}{6}}{2} + \binom{12}{6}}{2^{12}} = \boxed{\frac{1255}{2048}}. \end{aligned}$$

28. Pick $x_1, x_2, ..., x_{99}$ arbitrarily, for a total of $\boxed{100^{99}}$ possibilities. Let the remainder of their sum is divided by 100 be S. Then x_{100} must be equal to $100 - S$, which is uniquely determined for each choice of $x_1, x_2, \cdots, x_{99}$.

29. There are $\binom{8}{2}$ ways of choosing where the 0's go. Then, there are $\binom{6}{2}$ ways of choosing where the 1 goes. Now, there are 4 spots left. One of the 2's has to go in the rightmost open spot in order to be pure. Then, there are three open spots left for 2, 7, and 7. There are $\binom{3}{2}$ ways of choosing spots for the 7's. Since these steps are done in order, invoking product rule yields $\binom{8}{2} \cdot \binom{6}{2} \cdot \binom{3}{2} = \boxed{1260}$.

30. $a = 0$ is a solution. Otherwise, we can divide by a to get $x^2 + (4a-2)x + (3a^2 + 2a - 4) = 0$. This must be a perfect square, or else there would be 2 solutions, so $(2a-1)^2 = 3a^2 + 2a - 4$, which gives $a = 1, 5$, so the solutions are $a = \boxed{0,\ 1,\ 5}$.

31. As $x^2 + x + 1 = 0$, $x^3 = 1$. Thus, $x^{200} = (x^{66})^3 \cdot x^2 = x^2$ and $x^{100} = (x^{33})^3 \cdot x = x$, so $x^{200} + x^{100} + 1 = x^2 + x + 1 = \boxed{0}$.

32. It is evident that the circle with radius X is surrounded by rest of the circles. Connecting the pairwise centers of the circles with known radius, the side lengths of the triangle formed are $16, 17, 17$. Dropping the altitude to the base of this isosceles triangle, the height of the triangle to the base is $\sqrt{17^2 - 8^2} = 15$ by Pythagoras. Connect the vertices of the triangle to the center of the circle with radius X. By using pythagorean theorem, we know that the $\sqrt{(8+X)^2 - 8^2} + X + 9 = 15$. Simplifying this formula gives $\sqrt{X^2 + 16X} = 6 - X$. Squaring both sides yields $X^2 + 16X = X^2 - 12X + 36 \implies 28X = 36 \implies X = \boxed{\frac{9}{7}}$.

33. Let X denote the event that the ball selected is blue and let Y denote the event that the ball selected is from Bag B. By definition of conditional probability,

$$\Pr[Y|X] = \frac{\Pr[X \text{ and } Y]}{\Pr[X]} = \frac{\frac{1}{2} \cdot \frac{4}{7}}{\frac{1}{2} \cdot \frac{3}{8} + \frac{1}{2} \cdot \frac{4}{7}} = \boxed{\frac{32}{53}}.$$

34. Let the center of the circle be O. WLOG, we consider $\triangle OPQ$. Let the foot of the perpendicular from O to BC be M, so $BM = \frac{1}{2}BQ = 3$. Since $OP = 5$, from right triangle OBM, we have $OM = 4$. Identically, if the feet of the perpendiculars from O to BC and CA are L and N, then $OL = ON = 4$. Thus, O is the incenter of $\triangle ABC$.

Let $AB = x$. Recall that the area of $\triangle ABC$ is rs, where r is the inradius and s is the semiperimeter, and is also $\frac{24x}{2} = 12x$. Thus,

$$[\triangle ABC] = rs = 4\frac{\left(24 + x + \sqrt{24^2 + x^2}\right)}{2} = 2\left(24 + x + \sqrt{576 + x^2}\right),$$
$$[\triangle ABC] = 12x,$$
$$2\left(24 + x + \sqrt{576 + x^2}\right) = 12x,$$
$$24 + x + \sqrt{576 + x^2} = 6x,$$
$$\sqrt{576 + x^2} = 5x - 24$$
$$576 + x^2 = 25x^2 - 240x + 576,$$
$$240x = 24x^2,$$
$$240 = 24x,$$
$$x = 10.$$

Hence, $AB = \boxed{10}$.

35. Let $AB = x$ and $CD = y$. We want to find the length of $MN = \frac{x+y}{2}$. Consider the line l through B parallel to AC. Let l intersect CD at E. Then, $BE = AC = 11$ and $CE = AB = x$. Since $AC \perp BD$ and $AC \parallel BE$, we also have $BE \perp BD$. Thus, by Pythagorean theorem on right triangle $\triangle EBD$, we have that $(x+y)^2 = 60^2 + 11^2 = 3600 + 121 = 3721 = 61^2$, so $x + y = 61$. Hence, $MN = \frac{x+y}{2} = \boxed{\frac{61}{2}}$.

36. Note that

$$\frac{2022!}{2017!} = 2018 \cdot 2019 \cdot 2020 \cdot 2021 \cdot 2022,$$

so $2020 \mid \frac{2022!}{2017!}$, which implies that $101 \mid \frac{2022!}{2017!}$, so

$$33,632,280,3AB,168,080 \equiv 0 \pmod{101}.$$

Since $100 \equiv -1 \pmod{101}$, this implies that $33,632,280,3AB,168,080$ is congruent to

$$3(-1)^8 + 36(-1)^7 + 32(-1)^6 + 28(-1)^5 + 3(-1)^4 + (\overline{AB})(-1)^3 + 16(-1)^2 + 80(-1)^1 + 80(-1)^0$$
$$\equiv 3 - 36 + 32 - 28 + 3 - \overline{AB} + 16 - 80 + 80$$
$$\equiv -\overline{AB} - 10 \equiv 0 \pmod{101}.$$

Thus, $\overline{AB} \equiv -10 \equiv 91 \pmod{101}$, and since A and B are digits, this implies that $\overline{AB} = 91$. Hence, $10A + B = \boxed{91}$.

37. We express the degree 4 polynomial $P(x)-2x$ as $(x-1)(x-2)(x-3)(x-4)$, so $P(5) = 4\cdot3\cdot2\cdot1+10 = \boxed{34}$.

38. First, you want to find slowest the eastbound train can be going for it to not crash with the northbound one. In order for that to happen, the eastbound train has to hit the tail of the northbound one. The northbound train will be completely clear of the intersection in: $(200+25)/50 = 9/2$ hours. This means that the head of the eastbound train has to be there at this time, which leads to the minimum speed being: $300/(9/2) = 200/3$. For the maximum speed, the northbound train has to hit the tail of the eastbound train. The time it will take for the tail of the eastbound train to reach the intersection is $(300+50)/s$, where s is the speed. We want this to equal $200/50$, the time it takes for the northbound train to reach the intersection. Therefore, solve, and get that s is $175/2$. Therefore, take their difference, and you get $\boxed{125/6}$.

39. Let E be the expected value of P. The expected value of any roll of the die is $\dfrac{1+2+3+4+5+6}{6} = \dfrac{7}{2}$, so $a = b = c = d = \dfrac{7}{2}$. It follows that $E(1) = 7, E(2) = \dfrac{49}{4}, E(3) = 7, E(4) = \dfrac{49}{4}$.

    Since P has degree at most 3, E does as well, so its third finite differences are constant. Note that its first finite differences are

$$E(2)-E(1), E(3)-E(2), E(4)-E(3), E(5)-E(4), ...,$$

    its second finite differences are

$$E(3)-2E(2)+E(1), E(4)-2E(3)+E(2), E(5)-2E(4)+E(3), ...,$$

    so its third finite differences are

$$E(4)-3E(3)+3E(2)-E(1), E(5)-3E(4)+3E(3)-E(2), ...$$

    Thus,

$$\begin{aligned} E(4)-3E(3)+3E(2)-E(1) &= E(5)-3E(4)+3E(3)-E(2), \\ E(5) &= 4E(4)-6E(3)+4E(2)-E(1) \\ E(5) &= 4\left(\frac{49}{4}\right)-6(7)+4\left(\frac{49}{4}\right)-7 \\ &= 49-42+49-7 = \boxed{49}. \end{aligned}$$

40. Define the function $f(x)$ to return the number of inversions in the sequence when the sequence has 2^x terms.

    Parsing the recurrence, we can see that the terms $a_{2^x+1}, a_{2^x+2}, \cdots, a_{2^{x+1}}$ is formed by taking the first 2^x terms of the sequence, reversing their order, and adding 2^{x-1}. By uniqueness of binary representation, all numbers are distinct. We shall now express $f(x+1)$ in terms of $f(x)$.

    Among the first half of the sequence $\{a_i\}_{i=1}^{2^{x+1}}$, by definition, there are $f(x)$ inversions. Now, note that if (a_i, a_j) was an inversion in the first half, $(a_{2^{x+1}-i+1}, a_{2^{x+1}-j+1})$ will not be an inversion (since the order was reversed), and vice versa. Therefore, among the second half, there are exactly $\dbinom{2^x}{2} - f(x)$ inversions.

    Finally, it can be seen that the numbers in the first half of the sequence are between 0 and 2^x-1 inclusive, so each number in the second half will be strictly greater than any number in the first half, yielding no inversions across the two halves of the sequence. We may thus conclude from the previous paragraph that $f(x+1) = f(x) + \dbinom{2^x}{2} - f(x) = \dbinom{2^x}{2}$, so we may conclude that $f(n) = \dbinom{2^n}{2} = 2^{n-1}(2^n-1) = \boxed{2^{2n-1}-2^{n-1}}$.