Joe Holbrook Memorial Math Competition

8th Grade Solutions

October 13, 2019

1. There are 7 days in a week. Thus the number of pounds of potatoes generated per day is $119/7 = \boxed{17}$.
2. Simply perform division. We find $\boxed{5.75}$.
3. Since 2017, 2018, and 2019 form an arithmetic progression, $2017 + 2018 + 2019 = 3 \cdot 2018$. Hence the answer is $\frac{3 \cdot 2018}{6} = \frac{2018}{2} = \boxed{1009}$.
4. It is given that 3 and 5 are both less than 4. Further, $4 < 1 < 2$. Thus, $\boxed{4}$ is the median.
5. Let x be the answer to this question. Then $x = 1009 + \frac{x}{2019}$, implying $x = \boxed{1009.5}$.
6. Let the three integers be $n-1, n, n+1$. Then their sum is $(n-1)+n+(n+1) = 3n = 15$, and hence the middle integer is $\frac{15}{3} = 5$. Since the three integers are 4, 5, and 6, their product is $4 \cdot 5 \cdot 6 = \boxed{120}$.
7. Note that $\langle 6 \rangle = 1+2+3 = 6$, hence $\langle\langle 6 \rangle\rangle = \langle 6 \rangle = 6$, and $\langle\langle\langle 6 \rangle\rangle\rangle = \langle 6 \rangle = \boxed{6}$. (Fun fact: in number theory, we call n a *perfect number* if $\langle n \rangle = n$.)
8. We use PEMDAS. We have $3 \circ 2 = 3^2 - 2^2 = 9 - 4 = 5$. This leaves us with $7 \circ 5$, or $7^2 - 5^2 = 49 - 25 = \boxed{24}$.
9. To calculate the exponent tower, we first calculate $1^9 = 1$, then $0^{1^9} = 0^1 = 0$, and finally $2^{0^{1^9}} = 2^0 = 1$. Hence, the original expression equals $1 \times 2018 - 2017 = 2018 - 2017 = \boxed{1}$, our answer.
10. Let x be the number of people who voted twice. Let y be the number of people who voted once. We get the system of equations $x + y = 2020$ and $2x + y = 2019 + 2018$. Solving this yields $x = \boxed{2017}$.
11. Since x must be real, $k \geq 0$. Since x must be an integer, k must be a square. There are $\boxed{10}$ squares from 0 to 99 inclusive.
12. Isolating the sum $a^4 + b^4$, we find $a^4 + b^4 = 97$. Clearly, a and b must be less than 4. Testing $a = 3$, we find $b^4 = 97 - 3^4 = 16 \implies b = 2$. By symmetry the pair $(a, b) = (2, 3)$ satisfies the equation. Either way, the answer is $a + b = \boxed{5}$.
13. Given $2^x = 25$, we raise both sides of the equation to the $\frac{1}{2}$ power, yielding $2^{\frac{x}{2}} = 5$. If we want $2^{\frac{x}{2}+3}$, then we should multiply both sides of the equation by $2^3 = 8$. Thus, $2^{\frac{x}{2}+3} = 5 \cdot 8 = \boxed{40}$.
14. *Method 1:* The sum of the coefficients of a polynomial $P(x)$ is simply $P(1)$. To see this, note that if $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, then

$$P(1) = a_n 1^n + a_{n-1} 1^{n-1} + \ldots + a_1 1 + a_0$$

$$= a_n + a_{n-1} + \ldots + a_1 + a_0,$$

precisely the sum of the coefficients of $P(x)$. Hence, the answer is $(2 \cdot 1 + 1)^3 = 3^3 = \boxed{27}$.

Method 2: Expanding the given polynomial, we find

$$(2x+1)^3 = (2x+1)^2(2x+1) = (4x^2 + 4x + 1)(2x+1)$$

$$= 8x^3 + 12x^2 + 6x + 1.$$

Thus the sum of the coefficients is $8 + 12 + 6 + 1 = 27$, as above.

15. Derek reaches the shore in $\frac{600}{75} = 8$ minutes, after which there are 2 people in the water. Sameer reaches the shore in $\frac{600}{60} = 10$ minutes, after which only Daniel is in the water. Thus, exactly 2 people are in the water for $10 - 8 = \boxed{2}$ minutes.

16. The product is divisible by $2 \cdot 5 = 10$, so the remainder is either 0 or 10. The remainder can't be 0, as the product is only divisible by one power of 2. Hence our answer is $\boxed{10}$.

17. Let c denote the number of chickens on the farm, and let p denote the number of pigs on the farm. Since there are 21 heads total, including Eric's, $c + p + 1 = 21$. Also, since there are 52 feet total, including both of Eric's, $2c + 4p + 2 = 52$. Solving this system, we find the answer of $c = \boxed{15}$ chickens.

18. One Jerry can solve 12 problems in 4 seconds, so it can solve 3 problems in 1 second. In 6 seconds, a Jerry can solve $3 \cdot 6 = 18$ problems, so 5 Jerries can solve $18 \cdot 5 = \boxed{90}$ problems in 6 seconds.

19. Note that a can be negative. Hence, we take a to be the most negative it can be, so long b remains even. The least possible value of b is 2, which corresponds to the most negative value of $a = -128$, yielding a sum of $-128 + 2 = \boxed{-126}$.

20. The maximum possible area of a convex polygon with a fixed perimeter occurs when the polygon is equilateral. Thus, the maximum area of $MARVINYU$ occurs when the octagon is equilateral with side length $24/8 = 3$. To quickly compute the area of $MARVINYU$, denote $P_1 = MA \cap RV$, $P_2 = RV \cap IN$, $P_3 = IN \cap YU$, and $P_4 = YU \cap MA$. Then $P_1P_2P_3P_4$ is a square with side length $3 + 3\sqrt{2}$. The area of octagon $MARVINYU$ is equal to the area of $P_1P_2P_3P_4$ minus four times the area of the right isosceles triangle P_1RA. Thus, our answer is

$$(3 + 3\sqrt{2})^2 - 4 \cdot \frac{1}{2} \cdot \left(\frac{3\sqrt{2}}{2}\right)^2 = \boxed{18 + 18\sqrt{2}}.$$

21. Recall that if a number is divisible by 9, the sum of digits of the number must be divisible by 9. As 9 divides x, 9 must divide $2+5+9+1+A+0+B = 17+A+B$, implying that $A+B$ leaves a remainder of 1 when divided by 9. Similarly, as 9 divides y, 9 must divide $1+0+2+4+2+A+B+C = 9+A+B+C$, implying that $A + B + C$ leaves a remainder of 0 when divided by 9. It follows that C must leave a remainder of 8 when divided by 9. As C is a digit between 0 and 9, C must equal $\boxed{8}$, our answer.

22. Let S, A, and H denote the number of students in the Science, Arts, and Humanities departments, respectively. Note that 4 must divide S, 3 must divide A, and 2 must divide H. Thus, the least possible values for S, A, and H are 4, 3, and 2, respectively. The given condition states that $\frac{S}{4} + \frac{A}{3} + \frac{H}{2} = \frac{S + A + H}{3}$. By inspection, the triple $(S, A, H) = (4, 3, 2)$ satisfies this condition, and hence the smallest possible number of students at Barry's College of Academics is $4 + 3 + 2 = \boxed{9}$.

23. The hamster's volume during summer is $\pi r^2 \cdot 2r = 2\pi r^3$. The hamster's volume during winter is $\pi(r + 1)^2 \cdot (2r - 1)$. As the hamster's volume is conserved, we have

$$2\pi r^3 = \pi(r + 1)^2 \cdot (2r - 1) \implies 2r^3 = (r + 1)^2(2r - 1)$$
$$\implies 2r^3 = 2r^3 + 3r^2 - 1 \implies 3r^2 = 1.$$

Solving, we find $r = \boxed{\frac{\sqrt{3}}{3}}$, as desired.

24. Since there are four seats around the table, Simon must sit across from Doug. Since Doug must sit next to Sumner, Sumner must either sit to the right or left of Doug. If Sumner sits to the right of Doug, then Andy must sit to the left of Doug, and similarly, if Sumner sits to the left of Doug, then Andy must sit to the right of Doug. These two cases are rotationally distinct, and hence there are $\boxed{2}$ ways to seat the four people.

25. Since we are asked to find the infinite geometric series $1 - x + x^2 - x^3 + x^4 - \ldots = \frac{1}{1+x}$, the answer to the question is $\frac{1}{1+x}$. However, we are also told that the answer to the question is x. Hence, $\frac{1}{1+x} = x \implies x^2 + x - 1 = 0 \implies x = \frac{-1 \pm \sqrt{5}}{2}$. Since $x \in (0, 1)$, the answer is $x = \boxed{\frac{-1 + \sqrt{5}}{2}}$.

26. Note that $V_1 = (0,0)$ and $V_2 = (0,4)$. To find the intersection points A, B, we solve $x^2 = 4 - x^2 \implies x^2 = 2 \implies x = \pm\sqrt{2}$. Then the intersections are $(-\sqrt{2}, 2)$ and $(\sqrt{2}, 2)$. Thus, A, B are $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$ in some order. Since these points form a rhombus by symmetry, we can simply find the length of one of the sides and multiply by 4. As the distance between $(\sqrt{2}, 2)$ and $(0,0)$ is $\sqrt{2+4} = \sqrt{6}$, the answer is $\boxed{4\sqrt{6}}$.

27. For the distance between the clock hands to equal $17 = \sqrt{8^2 + 15^2}$, the smaller angle between the clock hands must equal 90 degrees. The earliest time the clock could form a 90 degree angle would be before 2 o'clock, some $m < 60$ minutes after 1 o'clock. In this case, the hour hand would be located at an angular position of $30 + \frac{m}{60} \cdot 30 = 30 + \frac{m}{2}$ degrees, while the minute hand would be located at an angular position of $360 \cdot \frac{m}{60} = 6m$ degrees. By inspection, the difference in these angular positions should be obtuse and hence equal 270, yielding $6m - \left(30 + \frac{m}{2}\right) = 270 \implies m = \frac{600}{11}$. Thus, it takes $\frac{600}{11} - 30 = \boxed{\frac{270}{11}}$ minutes for the hands to form 90 degrees.

   Fun fact: the (not necessarily acute or obtuse) angle between the hands of a clock at hour H, minute M is equal to $\frac{1}{2}|60H - 11M|$.

28. We have $A = 3 + \frac{1}{A}$, hence $A^2 - 3A - 1 = 0$. Solving gives us $A = \frac{3 \pm \sqrt{13}}{2}$. Since $\sqrt{13} > 3$, we must choose the positive solution, implying $A = \frac{3+\sqrt{13}}{2}$. Similarly, $B = \sqrt{3+B}$, hence $B^2 - B - 3 = 0$ and $B = \frac{1+\sqrt{13}}{2}$. Thus, $A - B = \boxed{1}$.

29. Notice that each of the coefficients of the polynomial, as well as the right hand side, are divisible by 10001. Dividing the equation by 10001, we are left with $2x^3 + x + 9 = 2019$. We recognize that using base 10 expansion, $2019 = 2 \cdot 10^3 + 1 \cdot 10 + 9$, and hence our answer is $x = \boxed{10}$.

30. By the Pythagorean Theorem on $\triangle ABC$, $AC = \sqrt{3^2 + 4^2} = 5$. By the Pythagorean Theorem on $\triangle CDA$, $DA = \sqrt{5^2 - 1^2} = 2\sqrt{6}$. Since $\triangle ABC$ and $\triangle CDA$ are right triangles, their areas are $\frac{AB \cdot BC}{2} = 6$ and $\frac{CD \cdot DA}{2} = \sqrt{6}$, respectively. Since the area of $ABCD$ equals the sum of the areas of $\triangle ABC$ and $\triangle CDA$, our answer is $\boxed{6+\sqrt{6}}$.

31. Note that the circle with diameter coinciding with the longest side contains $\triangle ABC$, as it is obtuse per the Pythagorean Inequality $(2^2 + 3^2 < 4^2.)$ Thus a circle of radius $\boxed{2}$ is achievable. Now we note that we can't do better because the circle must have diameter at least 4, since it contains a segment of length 4.

32. Note that for all $k \geq 0$, 2^{2k} leaves a remainder of 1 when divided by 3, while 2^{2k+1} leaves a remainder of 2 when divided by 3. Thus, $\left\lfloor \frac{2^{2k}}{3} \right\rfloor = \frac{2^{2k}-1}{3}$ and $\left\lfloor \frac{2^{2k+1}}{3} \right\rfloor = \frac{2^{2k+1}-2}{3}$, implying that $\left\lfloor \frac{2^{2k}}{3} \right\rfloor + \left\lfloor \frac{2^{2k+1}}{3} \right\rfloor = \frac{2^{2k}+2^{2k+1}}{3} - 1$ for all $k \geq 0$. Since the desired sum equals the sum of $\left\lfloor \frac{2^{2k}}{3} \right\rfloor + \left\lfloor \frac{2^{2k+1}}{3} \right\rfloor$ for $0 \leq k \leq 5$, it follows that the desired sum equals

$$\left(\frac{2^0+2^1}{3} - 1\right) + \left(\frac{2^2+2^3}{3} - 1\right) + \ldots + \left(\frac{2^{10}+2^{11}}{3} - 1\right)$$

$$= \frac{2^0 + 2^1 + \ldots + 2^{11}}{3} - 6 = \frac{2^{12}-1}{3} - 6 = \boxed{1359}.$$

33. The area of the parallelogram with vertices $(0,0), (x,1), (1,x)$, and $(x+1, x+1)$ is $x^2 - 1$. Thus,

$$\frac{1}{f(x)} = \frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right).$$

   Thus, the desired infinite sum equals

$$\frac{1}{2}\left(\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots\right).$$

This sum telescopes to $\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}\right)=\frac{1}{2}\cdot\frac{3}{2}=\boxed{\frac{3}{4}}$, which is the answer.

34. Let the stick be split into parts of length a and b meters. Then, by the given information, $a^2\pi + b^2\pi = 100\pi \implies a^2 + b^2 = 100$. Note that as $(a-b)^2 \geq 0$,

$$\begin{aligned} a^2+b^2 \geq 2ab &\implies 2(a^2+b^2) \geq (a+b)^2 \\ &\implies a+b \leq \sqrt{2(a^2+b^2)} \end{aligned}$$

with equality when $a = b$. It follows that the maximum value of $x = a + b$ is $\sqrt{2\cdot 100} = \boxed{10\sqrt{2}}$, achieved for $a = b = 5\sqrt{2}$.

35. We use complementary counting. The number of portions equals the total number of squares in a 5×5 grid minus the number of squares in a 5×5 grid that contain the center square. The number of squares in a 5×5 grid can be counted by casework: there are 5^2 1×1 squares, 4^2 2×2 squares, and so on, giving a total of $5^2+4^2+3^2+2^2+1^2 = \frac{5\cdot 6\cdot 11}{6} = 55$ squares. The number of squares in a 5×5 grid which contain the center square can also be counted by casework: 1 1×1 square, there are 4 2×2 squares, 9 3×3 squares, 4 4×4 squares, and 1 5×5 square. Thus, our answer is $55-(1+4+9+4+1) = 55-19 = \boxed{36}$.

36. Note that Bob's 3 card selection process is equivalent to simply choosing 3 cards from 20 to 50 inclusive at random. We calculate the desired probability by complementary counting: in other words, we will find the probability that all 3 cards have different tens digits.

There are 4 possible tens digits: 2, 3, 4, and 5. There are 10 cards with tens digit 2, 10 with tens digit 3, and 10 with tens digit 4, while there is only one with tens digit 5. If none of the 3 cards have tens digit 5, then there must be one with tens digit 2, one with tens digit 3, and one with tens digit 4: this can occur in $10\cdot 10\cdot 10 = 10^3$ ways. There are 3 ways in which one of the cards can have tens digit 5, and each of these cases has $10\cdot 10 = 10^2$ ways. As there are a total of $\binom{31}{3}$ selections of 3 cards, the complement of our desired probability is $\frac{10^3+3\cdot 10^2}{\binom{31}{3}} = \frac{260}{899}$, and hence our answer is $1-\frac{260}{899} = \boxed{\frac{639}{899}}$.

37. Let X be the intersection of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ so that $\triangle XCD$ is equilateral.

Since $ABCD$ is an isosceles trapezoid, it has a cirumcircle, i.e. D lies on the circumcircle on $\triangle ABC$. Now, as $\angle AED$ is an exterior angle to isosceles triangle $\triangle ECD$, $\angle AED = 2\angle ACD$. But $2\angle ACD = \angle AOD$, implying $\angle AED = \angle AOD$, or equivalently, that quadrilateral $AEOD$ is cyclic. Finally, calculating the power of the point X with respect to circle $AEOD$, we find

$$XE\cdot XO = XA\cdot XD \implies XO = \frac{5\cdot 15}{15\sqrt{3}/4} = \frac{20\sqrt{3}}{3} \implies EO = \boxed{\frac{35\sqrt{3}}{12}}.$$

38. The thing we are trying to minimize can be rewritten as $\sqrt{(a-3)^2+(b+2)^2}+\sqrt{(a+5)^2+(b+4)^2}$. This is equivalent to taking a point on a Cartesian plane and finding the sum of the distances from this point to the points $(3,-2)$ and $(-5,-4)$. The sum of these distances is clearly minimized when the point lies on the line segment between $(3,-2)$ and $(-5,-4)$, in which case the sum of distances is just the distance between the two points which is $\sqrt{2^2+8^2} = \boxed{2\sqrt{17}}$.

39. Instead of reflecting the ball once it hits a wall, we reflect the entire figure as shown below, noting that the distance the ball travels is the same in both situations.

Then the vertical distance traveled is 6 and the horizontal distance 5, yielding an answer of $\sqrt{5^2+6^2} = \boxed{\sqrt{61}}$ by the Pythagorean Theorem.

40. We notice that on the first round each coin will be flipped over. On the second round, half of them will be expected to be turned over. Now in each following round, the flipped over coins will be, in expectation, distributed between the heads and tails coins. Hence this ratio will be preserved throughout the process, and then at the end 1009 coins will be heads up.