Joe Holbrook Memorial Math Competition

8th Grade

October 18, 2020

General Rules

- You will have **75 minutes** to solve **40 questions**. Your score is the number of correct answers.
- Only answers recorded on the appropriate Google Form will be graded.
- You are to remain visible to your proctor at all times. Please have your video camera on during the exam.
- This is an individual test. Anyone caught communicating with another student or using technology in an inappropriate way will be removed from the exam.
- Scores will be posted on the website. Please do not forget your ID number, as that will be the sole means of identification for the scores.
- You may not use the following aids:
  - Calculator or other computing device
  - Compass
  - Protractor
  - Ruler or straightedge

Other Notes

- All answers are integers. Please enter them with no spaces in between into the Google Form. For a negative integer, please enter -7 not $- \ 7$.
- Do not include commas in your answers. For example, the number one thousand is to be entered 1000 not $1,000$.
- You must not write units in your answers.
- Ties will be broken by the number of correct responses to questions 31 through 40. Further ties will be broken by the number of correct responses in the last five questions.

1. Compute $\frac{5^2 \times 5^2}{5^2} + \frac{5^2 + 5^2}{5^2}$.

2. A pencil costs \$0.85 and an eraser costs \$0.92. You buy 7 pencils, which is 3 more than the number of erasers you buy. How many cents do you spend in total?

3. On Halloween, the sun sets at 5:52 pm. It will take an additional 87 minutes for full darkness to settle in. If Frank needs to be back when it turns completely dark and starts trick-or-treating at 4:37 pm, how many minutes will he have to trick-or-treat?

4. In Dougland there are three types of currency: the Suzy, the Snowman, and the Doug. Given that two Suzys are worth 5 Snowmans and 3 Snowmans are worth 14 Dougs. How many Dougs is 24 Suzys?

5. Jake has a pair of 6-sided dice with unique sides. The first die has sides $1, 3, 3, 4, 5, 6$ and the second die has sides $1, 2, 3, 4, 4, 6$. Jake roles both dice at the same time and takes the sum of the two dice. If $\frac{a}{b}$ is the probability of rolling a 7 in simplest terms, what is $a + b$?

6. In some strange world, dogs have $2x + 5$ toes on each leg and 3 legs, and cats have $3x - 3$ toes on each leg and 5 legs. In a room with 3 dogs and 2 cats, there are a total of 111 toes in the room. What is x?

7. Kevin refuses to go outside if the temperature has less than 5 factors. The temperatures this week were 61, 62, 63, 64, 65, 66, and 67 degrees. How many of these days did Kevin stay home?

8. Solve for x if $x^{\frac{5}{3}} = 32$.

9. Simon has a poster of dimensions 10 inches by 20 inches. He wants to scale it up proportionally so that its area grows by a factor of 72. It would have a width of $a\sqrt{b}$ inches and length of $c\sqrt{d}$ inches where both dimensions are in simplest form. What is $a + b + c + d$?

10. Evaluate
$$\lceil\sqrt{5}\rceil + \lceil\sqrt{6}\rceil + \lceil\sqrt{7}\rceil + \cdots + \lceil\sqrt{29}\rceil$$
Note: For a real number x, $\lceil x\rceil$ denotes the smallest integer that is greater than or equal to x.

11. Isaac and Renee were talking about their parents. Isaac asked Renee to guess their ages, so Renee asked for some hints. Isaac said, "my parents' ages each have either 3 or 4 factors." Renee said, "That's not enough information," so Isaac added, "My parents are both older than 20 and younger than 50, and their ages end in 9." Renee said again, "still not enough information," to which Isaac responded "My dad's at least one year older than my mom." What is the sum of the ages of Isaac's parents?

12. How many numbers less than 1000 are divisible by 6 and 4, but not 7?

13. Circle A has radius 6 and circle B has radius 4. They have a common external tangent of length 12. If the distance between their centers can be written in the form $a\sqrt{b}$ such that b is not divisible by the square of any primes, find $a + b$.

14. Quadrilateral $ABCD$ is inscribed in a circle with center O. E lies on minor arc CD such that $EO \perp CD$. If $\angle CED = 100^\circ$ and $\angle ADE = 90^\circ$, find $\angle ABC$.

15. There exists a nondegenerate triangle ABC with integer side lengths such that BD is an angle bisector, where D lies on $\overline{AC}$. BD cuts AC into pieces of lengths 7 and 3. Find the minimum possible perimeter of $\triangle ABC$.

16. A die is taken, and 4 of its edges are colored red, 4 are colored green, and 4 are colored blue. The die is then rolled, and the expected number of edges on the top face that are blue is in simplest terms $\frac{a}{b}$. What is $a + b$?

17. A group of n friends are playing a card game with a pack of 126 cards. After the cards are dealt out, everyone has at least one card, but the players realize that exactly one half of them each have one more card than each of the other half. What is the sum of the possible values of n?

18. A square of side length y is wholly inscribed within a square of side length x where x and y are integers. If the area between the two squares is 2020, find the minimum possible value of x.

19. How many permutations of the numbers 1, 2, 3, 4, and 5 are there such that no three consecutive numbers in the permutation form an arithmetic series?

20. Given the system of equations

$$xy = 6 - 2x - 3y,$$
$$yz = 6 - 4y - 2z,$$
$$xz = 30 - 4x - 3z,$$

find the positive solution of x.

21. Suppose m is a three-digit positive integer such that $\gcd(12, m) = 1$ and $12^{-1} \equiv 12^2 \pmod{m}$. What is m? (Note: $a^{-1} \pmod{b}$ is defined as the number k such that $a \cdot k \equiv 1 \pmod{b}$).

22. Call a word formed from the letters a, b, and c *mayan* if between any two a's (not necessarily adjacent), there is a b, between any two b's (not necessarily adjacent), there is a c, and between any two c's (not necessarily adjacent), there is an a. How many mayan words of length 2020 start with abc?

23. Point A is $(2\pi, -\pi)$, point B is $(\sqrt{7}, \sqrt{14})$, and point C is $(2\sqrt{7}, \sqrt{14} - \frac{\sqrt{7}}{2})$. The area of triangle ABC can be written as $\frac{a + b\sqrt{c}}{d}$, where a, b, c, d are integers, $\gcd(a, b, d) = 1$, and c is square-free. Find $a + b + c + d$.

24. Pick a point D on segment $\overline{BC}$ of non-degenerate $\triangle ABC$ such that $\triangle ABD$ is similar to at least one of $\triangle ACD$, $\triangle ADC$, $\triangle CAD$, $\triangle CDA$, $\triangle DAC$, and $\triangle DCA$. What is the sum for all the possible distinct values for $\angle ADC$?

25. Anthony, David, and Erez are talking to each other while respecting social distancing. Each of them is exactly 6 feet away from the other 2. Autumn does not want to be within 6 feet of Anthony, David, or Erez. The area, in square feet, of the space Autumn does not want to enter is $a\pi + b\sqrt{c}$, wherea, b, c are integers. Find $a + b + c$.

26. For how many values of c in the interval $[0, 1000]$ does the equation

$$7\lfloor x \rfloor + 2\lceil x \rceil = c$$

have a solution for x? Note: For a real number x, $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.

27. Compute the sum of all positive integers that are equal to 105 plus their largest prime factor.

28. A cube of side length $\frac{7}{3}$ is dropped into a random place on the 3D lattice in the coordinate space (a coordinate plane with an extra dimension), with the sides of the square parallel/perpendicular to the axes. The expected number of lattice points (points with integer coordinates) in the interior of the cube is $\frac{a}{b}$ in simplest terms, what's $a + b$?

29. Let triangle ABC be a triangle such that $\angle ABC = 120°$. Let D be a point on $\overline{AC}$ such that $\overline{BD}$ bisects $\angle ABC$. If the ratio of the length of $\overline{AB} : \overline{BC} = 2 : 1$, let the ratio of the length of $\overline{BD} : \overline{BC} = m : n$, where m and n are relatively prime integers. Find $m + n$.

30. A number with an even amount of digits is called a "sandwich number" if the first half of the digits are in strictly decreasing order and the second half of the digits are in strictly increasing order. For example, 987359 is a sandwich number while 988778 is not. If $\frac{m}{n}$ of all six-digit sandwich numbers are palindromes (numbers that read the same backward and forwards) and m and n are relatively prime positive integers, what is $m + n$?

31. Let $f(x)$ be a quadratic function with integer coefficients that satisfies $f(x) = x^2 + kx + k \cdot f(x - 1)$, where k is a positive integer. Compute $|f(1)|$.

32. Two boxes are stacked on top of each other, against a vertical wall. The bottom box comes out 12 inches, is 3 inches wide and 8 inches tall. The top box comes out 4 inches, is 3 inches wide and 6 inches tall. A 3-inch wide plank is secured so that it leans against these two boxes and the wall. Then the boxes are removed, while the plank stays still. What is the volume of the largest box that can fit under this plank, without poking out either side?

33. In a certain family, there is one father, one mother, and some children (some boys and some girls). The average number of male relatives for a person in the family (where the relatives are also in the family) is 6.75. If there are m males and f females in the family, compute the value of $10f + m$.

34. Yul is addicted to her phone. 20 minutes after she wakes up, she has a $\frac{1}{4}$ probability of picking up her phone. At every 20 minute interval, if she picked up her phone 20 minutes ago, the probability resets to $\frac{1}{4}$. Otherwise, the probability that she picks up her phone doubles. If Yul is continuously awake for 16 hours, what is the expected number of minutes it will take for her to pick up her phone 8 times?

35. Consider triangle ABC with circumcenter O, such that AO is parallel to BC. If the circumradius of ABC has length 4 and side BC has length 5, then compute the value of AC^2.

36. Let $A = \sqrt{-x^2 + 4x + 32}$ and $B = \sqrt{-x^2 - 4x}$, for a value of x such that both A and B are real. The maximum value of $A - B$ can be expressed as $\sqrt{n}$, where n is a positive integer. Compute n.

37. Let $f(x) = x\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor$. The sum of all positive x with $f(x) = 31$ is a/b, where a and b are positive integers. What is $a + b$? Note: For a real number x, $\lfloor x\rfloor$ denotes the largest integer that is less than or equal to x.

38. Compute the smallest positive integer b for which there exist positive integers A and B less than b such that the value of BA_{2b} (the two digit number BA in base $2b$) is 9 times the value of AB_b (the two digit number AB in base b).

39. Define a_n recursively such that $a_1 = 0$ and $n^2 a_{n+1} = (n+1)^2 a_n + 2n + 1$ for integers $n \geq 1$. Compute a_{50}.

40. How many integers $1 < n < 100$ have the property that there exists an integer $k > 1$ such that for all integers x and y, $x^k + y^k \equiv (x+y)^k \pmod{n}$?