Joe Holbrook Memorial Math Competition

4th Grade Solutions

March 20, 2022

1. A clown number must be a multiple of 15. We can try the first few multiples of 15 and see if we get a clown number.
$$15 : 1^2 + 5^2 = 26$$
$$30 : 3^2 + 0^2 = 9$$
$$45 : 4^2 + 5^2 = 41$$
Since 41 is prime, we have that the smallest clown number is $\boxed{45}$.

2. We use the equation: rate $\cdot$ hours = logs. For this problem, the rate is proportional to the number of woodchucks so we can rewrite the equation as woodchucks $\cdot$ hours = logs. We know have two equations:
$$6 \times 16 = 2$$
$$X \times 10 = 5.$$
Dividing the two equations and cross multiplying we get $X = \boxed{24}$.

3. At each intersection of two streets, two equal obtuse angles and two equal acute angles are formed. Therefore, exactly half the corners are acute angles which is equivalent to $\boxed{50}$ percent.

4. For an integer to be divisible by 7, 11, and 13, it must be divisible by their least common multiple, which is $7 \cdot 11 \cdot 13 = 1001$. The 4-digit integers which are multiples of 1001 are $1001, 2002, \cdots, 9009$. There are 9 of these. In total, there are $9999 - 1000 + 1 = 9000$ 4-digit numbers. Thus the probability that one is divisible by 7, 11, and 13 is $\frac{9}{9000} = \frac{1}{1000}$, which gives us an answer of $1 + 1000 = \boxed{1001}$

5. There are 7 consonants and 3 vowels. Because the first and last letters are consonants, the 2nd and 4th letters must be vowels (as there must be at least one vowel between two consonants.) The middle letter can be either a vowel or a consonant for the word to still work, so there are $7 \cdot 7 \cdot 3 \cdot 3 \cdot 10 = \boxed{4410}$ options.

6. Denote Moana's speed as m and the wind's speed as w. Since distance is equal to speed times time. $21 = (m + w) \times 3$ and $21 = (m - w) \times 4$. If the system of equations is solved, it can be found that $w = \frac{7}{8}$. So, the answer is $7 + 8 = \boxed{15}$.

7. We convert 111000 to base 10 because we know that the binary number is of the form 111abc. This is equal to $32 + 16 + 8 = 56$. We can add at most 7 to this number to keep the password 6-digits long (the largest 6-digit binary password possible is 63 in base 10). The largest prime number less than 63 is 61. In binary, $61_{10} = 111101_2$.Thus the last three digits are $\boxed{101}$.

8. The shortest possible value for XY is achieved when XY is parallel to AB, with a length of 5. The longest possible value for XY is achieved when $X = B$ and $Y = D$ or $X = C$ and $Y = A$, with a length of $\sqrt{5^2 + 9^2} = \sqrt{106}$. So, the longest integer length is 10. There are $10 - 5 + 1 = \boxed{6}$ integer values.

9. To play optimally, Team A wants to take out the players who have a higher chance of hitting their shots, so it is optimal for player 1 to shoot player 2. On Team B's turn, players 3 and 4 try to shoot at 1. The chance they both miss is $\frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$. 1 then shoots 3, which is the optimal play, since 3 has a higher chance of hitting their target than 4. 4 then shoots at 1 and has a $\frac{3}{4}$ chance of missing. Then 1 can shoot 4, ending the game with team A winning. The chance this all occurs is
$$1 \cdot \frac{1}{2} \cdot 1 \cdot \frac{3}{4} \cdot 1 = \frac{3}{8},$$
so the answer is $3 + 8 = \boxed{11}$.

10. Let D be the distance the train travels, s be the speed of the train, and t be the time it takes for the train to arrive. Clearly, $D = st$. Now let D', s', and t' be the pigeon's distance travelled, speed, and travel time respectively. Because the pigeon travels at a constant speed, we know $D' = s't'$. We want $D' = s't' \geq 2D = 2st$. We know that $t' = t$; the pigeon will stop flying when David reaches Autumn. Thus, $s' \geq 2s$. The pigeon must fly at least $\boxed{2}$ times as fast to have a travel distance twice as large.

11. Suppose Jim keeps x apples for himself. Then, also give x apples to each of his friends. There are then $8-5x$ apples left, which must be distributed amongst the four friends. This can be done $\binom{11-5x}{3}$ ways by Stars and Bars. x can only be 0 or 1 (there are not enough apples for higher values), so our answer is $\binom{11}{3} + \binom{6}{3} = 165 + 20 = \boxed{185}$.

12. We know that both the numerator and denominator are integers, and that the denominator must divide the numerator for the fraction to be an integer. We have $(n+3) \mid (3n+14)$. We want to cancel out the n term on the right. We can do this by subtracting 3 times $(n+3)$ since $(n+3)$ will still divide $3n + 14 - 3(n+3)$. This gives us that $(n+3) \mid 5$. The factors of 5 are $-5, -1, 1, 5$. Since we can have $n + 3 = -5, -1, 1, 5$, we find that the solutions of n are $n = -8, -4, -2, 2$. This gives us a sum of -12, and the absolute value of this is $\boxed{12}$.

13. Each set of four colors can correspond to exactly two different paintings. To see this, let's label the colors 1 to 5; consider a tetraherdon painted with colors 1, 2, 3, 4. Imagine placing the tetrahedron on the table so that corner 4 is pointed up. Corners 1, 2, 3 can either be clockwise or counterclockwise, and no rotation can change between a clockwise or counterclockwise tetrahedron; this is called chirality. There are $\binom{5}{4} = 5$ different possible sets of four paints to use, and each set of four paints leads to two paintings, so the answer is $2 \cdot 5 = 10$.

14. Label the midpoint of BC as M. Also, draw the altitude from point A to BC and mark the intersection as H. From the Pythagorean theorem, we know that $11^2 - CH^2 = AH^2$ and that $9^2 - (10 - CH)^2 = AH^2$. Therefore, $11^2 - CH^2 = 9^2 - (10 - CH^2)$. Now we can solve for CH: $121 - CH^2 = -19 + 20CH - CH^2$, $CH = 7$. Since $MC = 10/2 = 5$, that means $HM = 2$. We also know that $AH = 6\sqrt{2}$. Therefore, we can solve for BC^2: $BC^2 = (2^2 + (6\sqrt{2})^2)^2 = \boxed{76}$

15. We can rewrite the expression in progressive steps until we use only $x + y$ and xy:

$$\begin{aligned}
x^4y + x + y + xy^4 &= x + y + xy(x^3 + y^3) \\
&= x + y + xy(x^3 + 3x^2y + 3xy^2 + y^3 - 3x^2y - 3xy^2) \\
&= x + y + xy((x+y)^3 - 3xy(x+y)) \\
&= 5 + 3(5^3 - 3 \cdot 3 \cdot 5) = \boxed{245}
\end{aligned}$$

16. Note that Betty will draw the ace of spades on some random turn between 1 and 52, inclusive. If the ace is at position n (such that Betty will draw it on turn n), the chance that Betty wins is $\frac{1}{2^{n-1}}$, as Abby must get tails on every turn before then. Therefore, our probability is $\frac{1}{52} \cdot \frac{1}{1} + \frac{1}{52} \cdot \frac{1}{2} + \frac{1}{52} \cdot \frac{1}{4} + \cdots + \frac{1}{52} \cdot \frac{1}{2^{51}} = \frac{1}{52} \cdot (1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{51}})$. The term inside the parentheses is a geometric series and is very close to 2 (it adds up to $2 - \frac{1}{2^{51}}$), so $P \approx \frac{1}{52} \cdot 2 = \frac{1}{26}$, and $\frac{1}{P}$ is therefore very close to $\boxed{26}$.