Joe Holbrook Memorial Math Competition

6th Grade Solutions

March 20, 2022

1. One option is to just try all numbers starting with 2, until we find 19. Alternatively, notice that if x is divisible by 2 or 3, then $x+6$ will be as well, so the smallest prime factors of x and $x+6$ will be the same. So, we can instead restrict our search to x not divisible by 2 or 3 (equivalently, 1 or 5 mod 6). Then, we have to check $x = 1, 5, 7, 11, 13, 17$, until we find $x = \boxed{19}$, since the smallest prime factor of 19 is 19, while the smallest prime factor of 25 is 5.

2. Erez makes $60/10 = 6$ units per minute while Alicia makes $60/15 = 4$ units per minute, so together they make $4 + 6 = 10$ units per minute and it will take them $30/10 = \boxed{3}$ minutes to make the stellated icosahedron.

3. For an integer to be divisible by 7, 11, and 13, it must be divisible by their least common multiple, which is $7 \cdot 11 \cdot 13 = 1001$. The 4-digit integers which are multiples of 1001 are $1001, 2002, \cdots, 9009$. There are 9 of these. In total, there are $9999 - 1000 + 1 = 9000$ 4-digit numbers. Thus the probability that one is divisible by 7, 11, and 13 is $\frac{9}{9000} = \frac{1}{1000}$, which gives us an answer of $1 + 1000 = \boxed{1001}$.

4. If you drop the perpendicular from A onto CD (call the intersection point E), note that $ABCE$ is a rectangle, so $AE = BC = 15$, $AB = EC = 12$, and $DE = 20 - 12 = 8$. $\angle AED$ is a right angle, so by the Pythagorean Theorem, $AD = \sqrt{AE^2 + DE^2} = \sqrt{225 + 64} = \boxed{17}$.

5. We consider the different possible number of digits of the numbers less than 1000. A number less than 1000 either has 3, 2, or 1 digits.
   First, if the number has three digits, we know that the last digit is 9, and that the first two digits must be less than 9 since the digits are increasing. As a result, each of the first two digits is a digit from 1 to 8 inclusive. Neither digit can be 0; we don't want the first digit to be 0, and the second digit cannot be 0 because then there would be no possible digit for the first digit. Also, the two digits must be distinct because they must be increasing. Thus we must choose two numbers from 1 to 8. There are $\binom{8}{2} = 28$ ways to do this.

   Next, if the number has two digits, we know the last digit must be 9. Then the first digit is a digit from 1 to 8. There are 8 ways to choose the first digit.

   Lastly, if the number has one digit, the only possibility is the number 9.

   This gives us a total of $28 + 8 + 1 = \boxed{37}$ boom numbers less than 1000.

6. Since a, b, and 2021 are all positive integers, we know $a + b, a - b$ are both positive integers as well, with $a + b > a - b$. 2021 has two factor pairs (2021,1) and (47,43), so we can solve a system of equations for each pair: $a + b = 2021, a - b = 1 \implies a = 1011, b = 1010$, and $a + b = 47, a - b = 43 \implies a = 45, b = 2$. Thus, the sum of possible values of ab is $1011 \cdot 1010 + 45 \cdot 2 = \boxed{1021200}$.

7. The shortest possible value for XY is achieved when XY is parallel to AB, with a length of 5. The longest possible value for XY is achieved when $X = B$ and $Y = D$ or $X = C$ and $Y = A$, with a length of $\sqrt{5^2 + 9^2} = \sqrt{106}$. So, the longest integer length is 10. There are $10 - 5 + 1 = \boxed{6}$ integer values.

8. Let D be the distance the train travels, s be the speed of the train, and t be the time it takes for the train to arrive. Clearly, $D = st$.
   Now let D', s', and t' be the pigeon's distance travelled, speed, and travel time respectively. Because the pigeon travels at a constant speed, we know $D' = s't'$. We want $D' = s't' \geq 2D = 2st$. We know that $t' = t$; the pigeon will stop flying when David reaches Autumn. Thus, $s' \geq 2s$. The pigeon must fly at least $\boxed{2}$ times as fast to have a travel distance twice as large.

9. Nikhil's original chance of winning was $\frac{1}{2}$ because it was a fair game by symmetry (for example, getting 5 heads has the same probability as getting 0 heads). However, by Jaiden rigging the coin, Nikhil's chance of winning is now

$$\binom{5}{0}\left(\frac{1}{4}\right)^5+\binom{5}{1}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^1+\binom{5}{2}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2=\frac{1}{4^5}(1+15+90)=\frac{53}{512}.$$

Finally, $\frac{1}{2}-\frac{53}{512}=\frac{203}{512}\to m+n=\boxed{715}$.

10. If Nick swam for 10 minutes at 3 miles per hour, he swam $\frac{10}{60}\times 3=\frac{1}{2}$ a mile. If Marj has swam for 25 minutes at 4 miles per hour, she swam $\frac{25}{60}\times 4=\frac{5}{3}$ of a mile.
Call the distance Marj has left to swim after Marj and Nick meet m and the distance Nick has left to swim n. We know that $m+\frac{5}{3}-\frac{3}{4}=n+\frac{1}{2}$. Using power of a point, we also know that $\frac{5m}{3}=\frac{n}{2} \implies 10m=3n \implies n=\frac{10m}{3}$.
Finally, solving the system of equations, we have that, $m+\frac{5}{3}-\frac{3}{4}=\frac{10m}{3}+\frac{1}{2} \implies \frac{7m}{3}=\frac{5}{12} \implies m=\frac{5}{28}$. Therefore, Marj swims a total of $\frac{155}{84}$. This means $a+b=155+84=\boxed{239}$

11. We can first prime factorize 8! as $2^7\cdot 3^2\cdot 5\cdot 7$. We must choose exponents for a and b, for all four prime factors. First consider 2: The exponents of a and b, call them e and f, can be between 0 and 7, with $e\le f$. There are 8 ways to pick $e=f$ and $\binom{8}{2}=28$ ways to pick $e<f$. In total, there are 8^2 ways to pick e and f, so the probability that $e\le f$ is $\frac{8+28}{64}$. Repeating this for primes 3, 5, and 7, we get probabilities of $\frac{6}{9}$, $\frac{3}{4}$, and $\frac{3}{4}$. In total, our probability is then $\frac{36}{64}\cdot\frac{6}{9}\cdot\frac{3}{4}\cdot\frac{3}{4}=\frac{27}{128}$. The final answer is then $\boxed{155}$.

12. If the exponents of the primes in the prime factorization of n are $e_1, e_2, \ldots, e_k$, then we want

$$\frac{3e_1+1}{e_1+1}\cdot\frac{3e_2+1}{e_2+1}\cdots\frac{3e_k+1}{e_k+1}=7.$$

We can see that that $\frac{3e_i+1}{e_i+1}<3$. Further, when $e_i=1$, $\frac{3e_i+1}{e_i+1}=2$, so $2\le\frac{3e_i+1}{e_i+1}<3$. So, we must have $k=1$ or 2 (if we have at least 3, then our expression is at least $2^3=8$). If $k=1$, then $3e_1+1=7(e_1+1)$ yields no solutions. If $k=2$, then $(3e_1+1)(3e_2+1)=7(e_1+1)(e_2+1)$. Expanding both sides and rearranging, we get $2e_1e_2-4e_1-4e_2=6$. If we divide both sides by 2 and then add 4 to both sides, this allows us to factor (Simon's Favorite Factoring Trick) as $(e_1-2)(e_2-2)=7$. So, the only solutions are $e_1-2, e_2-2=1,7$. This gives $e_1=3, e_2=9$ (or vice-versa), and so the smallest possible value of n is $2^9\cdot 3^3=\boxed{13824}$.

13. If we have some integer $B_1B_2B_3\ldots B_{3k}$ in base 2, it is equivalent to $O_1O_2\ldots Ok$ in base 8 with $O_{i8}=B_{3i-2}B_{3i-1}B_{3i_2}$; something similar can be said for base 16, but with $4k$. The base 2 representation of 2022_{10} is 011111100110_2. Since 111111111111_2 is obviously a palindrome in base 8 and base 16, we know that our desired answer has at most 12 digits in base 2. Let $X_1X_2X_3X_4X_5X_6X_7X_8X_9X_{10}X_{11}X_{12}$ be our answer. To satisfy this being a palindrome in base 8, we must have $X_1X_2X_3=X_{10}X_{11}X_{12}$ and $X_4X_5X_6=X_7X_8X_9$. For this to be a palindrome in base 16, we must also have $X_1X_2X_3X_4=X_9X_{10}X_{11}X_{12}$. Experimenting with this and keeping in mind that it must be greater than $2022_{10}=011111100110_2$, we get an answer of $111101101111_2=3951_{10}$.

14. There's 5 ways to order 1 and 5 given the restrictions due to the vertical parts, namely,

$$51---$$
$$5-1--$$
$$-51--$$
$$-5-1-$$
$$15---$$

Now the path is simply horizontal lines, meaning every permutation of 2,3,4 is possible, for a total of $5 \cdot 6 = \boxed{30}$.

15. If the common difference d is not a multiple of 5, then one of the primes will be a multiple of 5. This is because for $a_{k+1} = a_1 + dk \equiv 0 \pmod 5$, we can set $k \equiv -a_1 \cdot d^{-1} \pmod 5$, for some value of k between 1 and 5. Similarly, d must be a multiple of 2 and 3, so d is a multiple of 30. Since we want the smallest possible value of p_6, we'll first consider $d = 30$. Obviously, we cannot have $a_1 = 2, 3, 5$, as then all of the primes will be multiples of 2, 3, or 5. If we set $a_1 = 7$, then the sequence is $7, 37, 67, 97, 127, 157$, all of which are prime, so the answer is $\boxed{157}$.

16. Since $AB \parallel CD$, we have that $\triangle ABE \sim \triangle CDE$. So, we can let $BE = 3k$ and $DE = 5k$. Then, since the diagonals are perpendicular, the area is $\dfrac{(3+5)(3k+5k)}{2} = 32k = 96$. So, $k = 3$, and $AB^2 = 3^2 + 9k^2 = \boxed{90}$.