Joe Holbrook Memorial Math Competition

7th Grade Solutions

March 20, 2022

1. Rachel travels $30 \cdot \frac{5}{3} = 50$ miles and Ross travels $\frac{2}{3} \cdot 45 = 30$ miles so they will be $\boxed{20}$ miles apart.

2. One option is to just try all numbers starting with 2, until we find 19. Alternatively, notice that if x is divisible by 2 or 3, then $x+6$ will be as well, so the smallest prime factors of x and $x+6$ will be the same. So, we can instead restrict our search to x not divisible by 2 or 3 (equivalently, 1 or 5 mod 6). Then, we have to check $x = 1, 5, 7, 11, 13, 17$, until we find $x = \boxed{19}$, since the smallest prime factor of 19 is 19, while the smallest prime factor of 25 is 5.

3. Let the speed of the lazy cats be L and the speed of the hard-working cats be H. Using (number of cats)(speed of each ca number of jobs we can write

$$\begin{aligned} 3S \cdot 10 &= 1 \\ 2F \cdot 5 &= 1. \end{aligned}$$

   Dividing the two equation gives us $3S = F$. We now want to find x such that $x \cdot S = 3 \cdot F$. Substituting $3S = F$ we get $x = \boxed{9}$ lazy cats.

4. By the Binomial Theorem, we have that $(x^3 + \frac{1}{3x^5})^8 =$

$$\sum_{k=0}^{8} \binom{8}{k} (x^3)^{8-k} (\frac{1}{3x^5})^k$$

   Therefore, by exponent rules, we want $3(8-k) = 5k$. Solving gives us $k = 3$, so our constant term is $\binom{8}{3} \cdot (\frac{1}{3})^3 = 56 \cdot \frac{1}{27} = \frac{56}{27}$. So, our final answer is $56 + 27 = \boxed{83}$.

5. It can be observed that $J = n + 9$, $H = n + 7$, $M = n + 12$, and $C = n + 2$, because J is the 10th letter in the alphabet, H is the 8th letter, M is the 13th letter, and C is the third letter. Then,

$$J + H + M + M + C = (n+9) + (n+7) + (n+12) + (n+12) + (n+2) = 5n + 42.$$

   Thus, we need for $5n + 42 \equiv 0 \pmod{2021}$, which is equivalent to saying that $5n \equiv 1979 \pmod{2021}$. Therefore, $n = \boxed{800}$.

6. Nikhil's original chance of winning was $\frac{1}{2}$ because it was a fair game by symmetry (for example, getting 5 heads has the same probability as getting 0 heads). However, by Jaiden rigging the coin, Nikhil's chance of winning is now

$$\binom{5}{0}\left(\frac{1}{4}\right)^5 + \binom{5}{1}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^1 + \binom{5}{2}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 = \frac{1}{4^5}(1 + 15 + 90) = \frac{53}{512}.$$

   Finally, $\frac{1}{2} - \frac{53}{512} = \frac{203}{512} \to m + n = \boxed{715}$.

7. Suppose that at a given time as the alarm is buzzing, m is the expected number of minutes until Aminah gets up (assuming she is still asleep). There is a probability of $\frac{1}{20}$ that she will get up at that moment, or in 0 minutes.

   Meanwhile, there is a probability of $\frac{19}{20}$ that she will hit the snooze button, in which case she will return to the original state where the alarm buzzes. Because the situation is identical, she will get up an expected

m minutes after that. In this case, Aminah will get up in $m+1$ minutes, where the 1 accounts for the minute she snoozed.

Because these two cases account for the only paths Aminah can take, we conclude $m = \frac{1}{20}\cdot 0 + \frac{19}{20}\cdot(m+1)$. Solving this equation, we find that $m = 19$. The case that we have worked with is equivalent to the initial situation where 0 minutes have passed, so $m = n = \boxed{19}$.

8. We will label the circle of radius 1 as O_4. Suppose the centers of O_1, O_2, O_3, and O_4 are at W, X, Y, and Z respectively, and the intersection points of O_3 with O_1, O_2, and O_4 are at L, M, and N respectively. Because O_1 and O_2 both have diameter $2r$, $\overline{LM}$ is the diameter of O_3. Because W is the center of O_1 and $\overline{YZ}$ is tangent to O_1, $m\angle WYZ = m\angle XYZ = 90^\circ$. Hence, $\triangle WYZ$ is right. Applying the Pythagorean Theorem, we find that $r^2 + YZ^2 = (r+1)^2$. Moreover, YN and YL are radii of O_3, so $YN = YZ + 1 = 2r = YL$. Thus, we can substitute $YZ = 2r - 1$. All terms in the Pythagorean relationship are now expressed in r:

$$r^2 + (2r-1)^2 = (r+1)^2$$

$$r^2 + 4r^2 - 4r + 1 = r^2 + 2r + 1$$

$$4r^2 - 6r = 0$$

We select the positive value for r and find that $r = \frac{3}{2}$. The area of O_4 is thus $(\frac{3}{2})^2\pi = \frac{9}{4}\pi$, so $a + b = 9 + 4 = \boxed{13}$.

9. We can rewrite the parabola as $y = -4(x-6)^2 + 30$ to find that the vertex of the arc is $(6, 30)$. Assuming that one of the vertices of the banner that is on the parabola is (x, y), we know that $y = 2(x-6)$ since the height of the banner is equal to the width. Now we have a system of equations to solve. If we plug in $y = 2x - 12$ into the parabola equation we have, $0 = -4x^2 + 46x - 102 = 2x^2 - 23x + 51$. This can be factored to $0 = (2x-17)(x-3)$, so $x = \frac{17}{2}$ or 3. Plugging the x-values back into either original equation we also know that $y = -6$ or 5. Since the banner cannot go below the ground, we know that the banner has a side length of 5. Therefore, the area is $5^2 = \boxed{25}$.

10. Consider the integers 1 to 51 in mod 17; we end up with 3 of each integer from 0 to 16. There are 17^2 total pairs (x, y) possible, and 17 of these fit the restriction that $x + y = 0$ or 17 (every x has a corresponding y). This means $\frac{m}{n} = \frac{17}{17^2} = \frac{1}{17} \implies \boxed{18}$.

11. The greatest integer less than 2000 with only even digits is 888. Similarly, the smallest integer greater than 2000 with only even digits is 2002. Therefore, the input 2000 could represent any integer in the interval $[\frac{2000+888}{2}, \frac{2000+2002}{2}]$, or $[1444, 2001]$.
Similarly, the greatest integer less than 600 with only even digits is 488, and the smallest integer greater than 600 with only even digits is 602. Therefore, the input 600 represents could be any integer in the interval $[\frac{600+488}{2}, \frac{600+602}{2}]$, or $[544, 601]$.
By the same reasoning, 40 could represent any integer in the interval $[\frac{40+28}{2}, \frac{40+42}{2}]$, or $[34, 41]$, and 8 could represent any integer in the interval $[7, 9]$.
Choosing 1444, 544, 34, and 7 as our intended inputs makes our displayed result as far from the intended result as possible. Thus, our maximum error is $(2000 - 1444) + (600 - 544) + (40 - 34) + (8 - 7) = 556 + 56 + 6 + 1 = \boxed{619}$.

12. Since $AB \parallel CD$, we have that $\triangle ABE \sim \triangle CDE$. So, we can let $BE = 3k$ and $DE = 5k$. Then, since the diagonals are perpendicular, the area is $\frac{(3+5)(3k+5k)}{2} = 32k = 96$. So, $k = 3$, and $AB^2 = 3^2 + 9k^2 = \boxed{90}$.

13. If the common difference d is not a multiple of 5, then one of the primes will be a multiple of 5. This is because for $a_{k+1} = a_1 + dk \equiv 0 \pmod 5$, we can set $k \equiv -a_1 \cdot d^{-1} \pmod 5$, for some value of k between 1 and 5. Similarly, d must be a multiple of 2 and 3, so d is a multiple of 30. Since we want the smallest possible value of p_6, we'll first consider $d = 30$. Obviously, we cannot have $a_1 = 2, 3, 5$, as then all of the primes will be multiples of 2, 3, or 5. If we set $a_1 = 7$, then the sequence is $7, 37, 67, 97, 127, 157$, all of which are prime, so the answer is $\boxed{157}$.

14. Let E_k equal how many more lunchboxes Nikhil would, on average, need to get after already receiving k different types. We know that $E_0 = E_1 + 1$ because he is guaranteed to have gotten one type after getting his first one. $E_1 = \frac{2}{3}(E_2+1)+\frac{1}{3}(E_1+1)$ because he has a two thirds chance of his next lunchbox being a new type and a one third chance of it being the one he has already tried. $E_2 = \frac{1}{3}(E_3+1)+\frac{2}{3}(E_2+1)$ using similar reasoning. By definition, $E_3 = 0$; if he has found three different types, there is no need to get any more. We want to find E_0, so we plug in $E_3 = 0$ into our equation: $E_2 = \frac{1}{3}(0+1)+\frac{2}{3}(E_2+1) \rightarrow \frac{1}{3}E_2 = 1 \rightarrow E_2 = 3$. From this, $E_1 = \frac{2}{3}(3+1) + \frac{1}{3}(E_1+1) \rightarrow \frac{2}{3}E_1 = \frac{9}{3} \rightarrow E_1 = \frac{9}{2} \rightarrow E_0 = \frac{11}{2} \rightarrow m + n = \boxed{13}$.

15. We can plug in $x = \lfloor x \rfloor + \{x\}$ where $\{x\}$ represents the fractional part of x. (In other words, $\{x\} = x - \lfloor x \rfloor$.) With this substitution, we get

$$\begin{aligned} f(x) &= \lfloor \lfloor x \rfloor^2 + 2\lfloor x \rfloor \{x\} + \{x\}^2 \rfloor + \lfloor x \rfloor^2 \\ &= 2\lfloor x \rfloor^2 + \lfloor 2\lfloor x \rfloor \{x\} + \{x\}^2 \rfloor. \end{aligned}$$

If $\lfloor x \rfloor = 0$, there is 1 possible value of k. If $\lfloor x \rfloor = 1$, there are 3 possible values. If $\lfloor x \rfloor = 2$, there are 5 possible values. In general, for $\lfloor x \rfloor = n$, there are $2n - 1$ values of k.

50 is one more than the sum of the first seven positive odd integers. We can either find this by adding the numbers up or by noticing that the sum of the first n odd numbers is n^2. The 50th smallest value of k is the smallest value of k when $\lfloor x \rfloor = 7$. From here, the desired value is easily obtained by plugging in $x = 7$ in $f(x)$, which gives $2 \cdot 7^2 = \boxed{98}$.

16. We essentially want 1 intersection between the three chords. We can see that if there are no intersections, the three chords will only split the circle into four parts. We can casework on the number of points the three chords take up.

If the three chords take up four points, there are $\binom{10}{4}$ ways to pick the 4 points. There is one way to make the intersection and 4 ways we can attach the third chord, for a total of $\binom{10}{4} \cdot 4$.

If 5 points are taken up, there are $\binom{10}{5}$ ways to pick the 5 points. There are 5 ways to pick which of the four points should make up the two chords that intersect. There are two ways to make the final chord to the 1 unused point for a total of $\binom{10}{5} \cdot 10$.

If 6 points are taken up, there are $\binom{10}{6}$ ways to pick the 6 points. There are 6 ways to pick which four consecutive points should make up the two chords that intersect and 1 way for the final chord. (The two unused points making the final chord must be adjacent to prevent more intersections, which implies the four points making up the two intersecting chords should also be consecutive.) This is a total of $\binom{10}{6} \cdot 6$.

There are $\binom{10}{2} = 45$ pairs of distinct points (i.e., chords) for a probability of

$$\frac{\binom{10}{4} \cdot 4 + \binom{10}{5} \cdot 10 + \binom{10}{6} \cdot 6}{45^3} = \frac{308}{6075} \implies \boxed{6383}$$

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