Joe Holbrook Memorial Math Competition

4th Grade Solutions

October 16, 2022

1. The first five positive integers are $1, 2, 3, 4, 5$. We can find their average by taking their sum and dividing by 5; we get $\frac{15}{5} = \boxed{3}$.
2. The amount left will be the difference of 240 and 115, which equals $\$\boxed{125}$.
3. $20 \cdot 22 = \boxed{440}$.
4. Since $2 + 0 + 2 + 2 = 6$, our expression simplifies to $2^6 + 6^2 = 64 + 36 = \boxed{100}$.
5. Since Joh hates 3 classes, he likes the other $10 - 3 = 7$ classes. Liking $\frac{7}{10}$ classes is the same as $\boxed{70}\%$.
6. There are $3 \cdot 12 = \boxed{36}$ eggs.
7. For them to both make 1200 cutouts, Timmy takes $1200/10 = 120$ hours to finish, and Shawn takes $1200/15 = 80$. The answer is $\boxed{40}$.
8. For Nikhil to deadlift twice the weight of 50 pounds, he would be able to deadlift 100 pounds, which is $\boxed{95}$ pounds more than Lance.
9. Because a problem's difficulty is equal to the number of minutes it takes to write a solution for it, we can deduce that Lance's problems' average time to write solutions for is 6 minutes. So it'll take $12 \times 6 = \boxed{72}$ minutes for him to write all of his solutions.
10. The slice that Saria ate by herself was $\frac{1}{15}$ of the pizza, and half of the other slice that she ate was $\frac{1}{30}$ of the pizza. In total, the fraction of the pizza that she ate was $\frac{1}{15} + \frac{1}{30} = \frac{1}{10}$. This is $\boxed{10}\%$ of the pizza.
11. Eshaan needs to listen to songs for $\frac{293}{14}$ days, but since that doesn't divide evenly, we use the smallest integer that's higher than that number, which is $\boxed{21}$ days.
12. $17 \bullet (1 \bullet 1) = 17 \bullet 3 = 17 + 3 + 17 \cdot 3 = \boxed{71}$.
13. Rose should apply the 20% off coupon first because the base value it is being applied to is greater, meaning a greater value is being deducted. This coupon reduces the price to $100 \cdot 0.8 = 80$. After the second coupon, Rose only has to pay $80 - 10 = \boxed{70}$.
14. We plug in primes starting from the smallest odd value, 3, until the expression works. The first prime to work is $\boxed{11}$
15. The area of a square is equal to its side length squared. So a square with area 196 inches^2 has a side length of $\sqrt{196} = 14$ inches. A regular nonagon with three times this side length would have a side length of $14 \cdot 3 = 42$ inches. Since a regular nonagon has nine sides of equal length, the perimeter of this nonagon would be $9 \cdot 42 = \boxed{378}$ inches.
16. The most effective way to cut the cake is to start at a corner and cut each subsequent piece right next to the previous cut. This minimizes the number of leftover pieces of cake that are too small to be turned into a 2" x 2" piece. However, notice that following this method, there will always be a 1" x 4" strip leftover which cannot be turned into any pieces. Therefore, only the 12" x 4" portion is usable, giving us $\frac{12 \cdot 4}{2 \cdot 2} = \boxed{12}$ pieces.

17. We attempt to quickly narrow the possibilities by choosing the most limiting choices. Suppose the number is a single digit, this narrows the list down to $1, 2, 3, 4, 5, 6, 7, 8, 9$. Suppose the number is prime, giving the list $2, 3, 5, 7$. Finally, suppose the number is not odd, or even, giving an answer of $\boxed{2}$.

18. Average speed is equal to total distance divided by total time. Assume the total distance of the race is x miles long. It takes Pablo $\frac{\frac{x}{3}}{60}$ to ride the first third of the race, $\frac{\frac{x}{3}}{30}$ to ride the second third of the race, and $\frac{\frac{x}{3}}{55}$ to ride the last third of the race (since time is equal to distance divided by speed). In total, it takes Pablo $\frac{x}{180} + \frac{x}{90} + \frac{x}{165} = \frac{x}{44}$ hours to run the race. Therefore, Pablo's average speed is $\frac{x}{\frac{x}{44}} = \boxed{44}$ mph.

19. Call the length of the picture frame l and the width of the picture frame w. We are given that $2l+2w = 66$ (where $2l+2w$ is the perimeter of the frame) and that $l = 2w$. We can plug $l = 2w$ into the first equation to obtain $6w = 66$. So, $w = 11$ and $l = 22$. Since there is a 2 inch border on all sides, the length of the actual picture is $l - 4 = \boxed{18}$.

20. Let the number of vowels be x, the number of consonants would then be $x+4$, this means that $x(x+4) = 252$. By factoring 252, we find that a set of factors is 14 and 18, which the product is 252. This means there are $\boxed{14}$ vowels in Jacob's alphabet.

21. If a has 9 positive multiples less than 100, then $\frac{100}{a}$ must be equal to 9 with some remainder or 10 with no remainder (since we are only counting multiples LESS than 100, not equal to 100). Using the fact that the quotient decreases as the divisor increases for positive numbers and a constant positive dividend and that $100/9 = 11$ w/remainder, $100/10 = 10$ w/no remainder, $100/11 = 9$ w/remainder, and $100/12 = 8$ w/remainder, a can only be 10 or 11. If b has 19 positive multiples less than 100, then $\frac{100}{a}$ must be equal to 19 with some remainder or 20 with no remainder. Since $100/4 = 25$ w/no remainder, $100/5 = 20$ w/no remainder, and $100/6 = 16$ w/remainder, b can only be 5. Thus, ab is either 50 or 55. However, in either case scenario, there is only 1 multiple of ab strictly less than 100 and that is $ab \cdot 1$. Therefore, $\boxed{1}$ is our answer.

22. Solving the equation, we see that $x = -\frac{k}{3}$, and $-\frac{k}{3} < -\frac{2}{5}$ occurs exactly when $\frac{6}{5} < k$. Thus, the desired probability is $\frac{8\frac{4}{5}}{10} = \frac{22}{25}$ implies $\boxed{47}$.

23. A car traveling at 60 mph for 450 miles takes $\frac{15}{2}$ hours. A plane traveling at 240 mph for 400 miles takes $\frac{5}{3}$ hours, and adding two hours from the overhead gives $\frac{11}{3}$ hours. The difference in hours is then $\frac{23}{6}$, which is equivalent to $\boxed{230}$ minutes.

24. We note that for any fishy number n, n cannot be a one digit number: if so, then because its only digit would be prime and n itself would be prime, contradicting the problem. Therefore, $n \geq 10$. Both of n's digits must be elements of $\{0, 4, 6, 8, 9\}$. Since $n > 2$, n cannot be even as this would make n composite, so the units' digit of n must be 9, making the only possible fishy numbers $49, 69, 89$, and 99. Because $49 = 7^2, 69 = 3 * 23$, and $99 = 3^2 * 11$, the prime number 89 is the only fishy number, so the sum is $\boxed{89}$.

25. Due to the crust, the part of the pizza covered with cheese and pepperoni is actually a 14 inch diameter circle. This means that, if we were to ignore the pepperoni slices, Tyrone would need $7^2\pi = 49\pi$ square inches of cheese. However, the 20-slices of pepperoni take up $20 \times 1^2\pi = 20\pi$ square inches. Therefore, the area that the cheese needs to cover is $49\pi - 20\pi = 29\pi \approx \boxed{87}$ square inches.

26. We find out that the distance between points A and B have a change of 6 in x and 8 in y, which can be used with the Pythagorean Theorem to get a distance of 10. If a semicircular path was taken, then it would have diameter 10, and circumference of 10π. This means the semicircular path would have length 5π. The difference of these is $5\pi - 10$, giving an answer of $\boxed{15}$.

27. We notice the pattern that the unit digit of powers of 7 repeats as 7, 9, 3, and 1. We also see that the unit digit of powers of 2 repeats in 2, 4, 8, and 6. Therefore 9+4=13, so the answer is $\boxed{3}$.

28. We know that if a convex polygon has n sides, then its interior angle sum is given by $(n-2) * 180°$. We also know that because it is convex, each angle is less than 180 degrees. So, we only need to find the closest multiple of 180 greater than 2022. $180 * 12 = 2160$, so our answer is $180 * 12 - 2022 = \boxed{138}$.

29. Depending on how x compares to 5 and 8, the median must be 5, 8, or x. This means the mean, equal to $\frac{57+x}{9}$, must be 6, 9, or $x+1$. Solving for x, we get possible solutions of -3, 24, and 6, respectively; all of these end up working so the answer is $-3+24+6=\boxed{27}$.

30. We know that Austin's code is in the form $27ABC$ where A, B, C represent the digits of the code. Since the code is divisible by both 5 and 4, we know that $C=0$. Since the code is divisible by 9, we know that the sum of the digits of the code is divisible by 9 i.e. $2+7+A+B+C=9$ or 18 or 27. This means, $A+B=9$ or 18. Since the code is divisible by 8, the number ABC is divisble by 8. This leaves a limited number of options for $AB: 04, 08, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92, 96$. Of these possible values of AB, we can count that there are $\boxed{2}$ which satisfy the fact that $A+B=9$ or 18. Specifically, 36 and 72.

31. The smallest number n such that $s(n)=k$ will have approximately $k/9$ digits, so a larger k means a larger n. The smallest integer whose sum of digits is eleven is 29, and the smallest integer whose sum of digits is 29 is $\boxed{2999}$ (we minimize the number of digits by using many 9s, then put the smallest digit in the largest place value.)

32. Let O be the center of the semicircle. By symmetry, $OC=OD=\frac{6}{2}=3$, and by the Pythagorean Theorem, the radius OB equals $\sqrt{7^2+3^2}=\sqrt{58}$. The area of the semicircle is therefore $\frac{1}{2}\left(\sqrt{58}\right)^2\pi=29\pi$, so $a=\boxed{29}$.

33. The total number of ways to arrange RROOYY is $\frac{6!}{2!2!2!}=90$, because there are 2 of each letter. Then, we can use PIE to find the total number of ways there will be at least one ROY. Using ROY as a single unit, there are 4! ways to arrange the [ROY], R, O, Y. However, we have counted ROYROY twice, so we subtract one of them to get rid of the overcounting. Thus, the fraction is $\frac{4!-1}{90}=\frac{23}{90}$, so the answer is $\boxed{67}$.

34. The probability of there being exactly k heads is $\binom{5}{k}(\frac{3}{4})^k(\frac{1}{4})^{5-k}$. This is because there are $\binom{5}{k}$ ways to choose which k coin flips give heads, and then a probability $(\frac{3}{4})^k(\frac{1}{4})^{5-k}$ that the chosen coins give heads while the others give heads. So, the probability that Odd-Todd will win is $(\frac{3}{4})^3(\frac{1}{4})^2\binom{5}{3}+(\frac{3}{4})^5\binom{5}{5}=\frac{513}{1024}$, meaning the probability that Even-Steven wins is $1-\frac{513}{1024}=\frac{511}{1024}$. So, the positive difference in their probabilities is $\frac{2}{1024}=\frac{1}{512}$ and our answer is $\boxed{513}$.

35. We know that N's second smallest factor is p, its smallest prime factor. Thus, N's second largest factor must be $\frac{N}{p}$. So, we are saying that $\frac{N}{p}=15p \implies N=15p^2$. What are the possible values of p? Since p is the smallest prime factor and N is already divisible by 3 and 5, p can only be 2 or 3. So, our final answer is $15\cdot 2^2+15\cdot 3^2=\boxed{195}$.

36. The chance that there is one wrong digit will be $\left(\frac{2}{3}\right)^3\frac{1}{3}$, but any of the four digits might be wrong, so there is a multiplicity of 4. The probability would then be $\frac{32}{81}$, so the answer is $\boxed{113}$.

37. Since we have 64 squares in total, there are $\binom{64}{2}=32\cdot 63$ ways to pick two squares (where the order that we pick the squares doesn't matter). In each of the 8 columns, there are 7 ways to pick two squares that are vertically adjacent (the top square can be any of the top 7 squares in the column). So, there are $7\cdot 8=56$ pairs of vertically adjacent squares. By symmetry, there are also 56 pairs of horizontally squares. In total, our probability is $\frac{56\cdot 2}{32\cdot 63}=\frac{1}{18}$. So, our final answer is $\boxed{19}$.

38. The amount of grass the goat can eat is equivalent to the area the rope can sweep while being tied to the building. We observe that the rope can first sweep a sector of a circle with radius 3, measuring $300°$. This initial area therefore has area $\frac{300}{360}*\pi*3^2=\frac{15\pi}{2}$. However, since the building only has side length 2, once

the rope sweeps such that it is adjacent to the walls, its length of 3 allows it to further sweep towards the back of the building in a sector with radius $3 - 2 = 1$ with a central angle of $180° - 60° = 120°$. This results in two additional areas on either side of the building, totaling to $2 * \frac{120}{360} * \pi * 1^2 = \frac{2\pi}{3}$. Summing the areas we've calculated, the sum $\frac{15\pi}{2} + \frac{2\pi}{3} = \frac{49\pi}{6}$ is achieved, giving an answer of $49 + 6 = \boxed{55}$.

39. Because Amir can only go to points (a, b) where at least one of $\{a, b\}$ is even, he must take two consecutive steps in a given direction in order to be able to move perpendicularly. Hence, we can consider his path as a sequence of 8 moves: 4 that consist of 2 consecutive steps right and 4 that consist of 2 consecutive steps up. These 8 moves can be arranged in any way to form a unique path: because the 4 right-steps are identical, and so are the 4 up-steps, the number of possible sequences is

$$\frac{8!}{4! \cdot 4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = \boxed{70}.$$

40. Because $O_1O_2 = 6\sqrt{3} < 2(6)$, the circles intersect at two points, which we will call I_1 and I_2. Denote the midpoint of $\overline{I_1I_2}$ as M. Because $O_1M + MO_2 = O_1O_2$, by symmetry $O_1M = MO_2 = 3\sqrt{3}$. We observe that the segments from each center to intersection point is a radius of a circle, so $O_1I_1, O_1I_2, O_2I_1, O_2I_2 = 6$. $3\sqrt{3} = \frac{6}{\sqrt{3}}$, so $\triangle MO_1I_1, \triangle MO_1I_2, \triangle MO_2I_1$, and $\triangle MO_2I_2$ are congruent $30 - 60 - 90$ right triangles. Hence, $I_1M = I_2M = 3$, so the area of each triangle is $(\frac{9}{2}) \cdot \sqrt{3}$. Furthermore, $m\angle I_1O_1I_2 = m\angle I_1O_2I_2$ $= 60°$, so because the total area of each circle is 36π, the areas of the sectors enclosed by arcs $I_1O_1I_2, I_1O_1I_2$ are both 6π. Again using the symmetry of the desired region, we see that the area of the circles' union can be represented as $2(6\pi - 2[\triangle MO_2I_1])$, which is $2(6\pi - 9\sqrt{3}) = 12\pi - 18\sqrt{3}$. Hence, $a + b + c = 12 - 18 + 3 = \boxed{-3}$.